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COURANT INSTITUTE OF MATHEMATICAL SCIENCES

Computer Science NSO-19

COMPUTING CHROMATIC POLYNOMIALS FOR SPECIAL FAMILIES OF GRAPHS

Beatrice M. Loerinc

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Abstract

Let $P(G;\lambda)$ represent the chromatic polynomial of graph G with n vertices. We present and implement a complete graph basis algorithm for generating the coefficients of $P(G;\lambda)$, utilizing Whitney's identity. This algorithm is compared with a null graph basis algorithm and a tree basis algorithm already in the literature, and shown to be more computationally efficient for dense graphs.

A graph G is chromatically unique if $P(G;\lambda) = P(H;\lambda)$ implies H is isomorphic to G. $\theta_{d,e,f}$, the connected graph consisting of three edge-disjoint paths of lengths $d \leq e \leq f$ between two vertices of degree three, is proven chromatically unique. Two graphs G and H are chromatically equivalent if $P(G;\lambda) = P(H;\lambda)$. Let $\bar{G}$ denote the complement of G. We show $P(\bar{\theta}_{2,e,f};\lambda) = P(\bar{\theta}_{2,e',f'};\lambda)$ whenever $e + f = e' + f'$. For 17 vertices or less, all $\bar{\theta}_{d,e,f}$, $d \neq 2$, are chromatically distinct, i.e., among all θ-graph complements, their chromatic polynomials are different.

It is well known that all trees on n vertices are chromatically equivalent. We show that tree complements for $n \leq 7$ are chromatically distinct, but for $n = 8, 9, 10$ and 11, pairs of nonisomorphic trees T_i and T_j are exhibited for which $P(T_i;\lambda) = P(T_j;\lambda)$ and $P(\bar{T}_i;\lambda) = P(\bar{T}_j;\lambda)$. This contradicts a privately communicated conjecture stating that if two graphs G and H are nonisomorphic, then $P(G;\lambda) \neq P(H;\lambda)$ or $P(\bar{G};\lambda) \neq P(\bar{H};\lambda)$ or both.

We find closed formulas for the chromatic polynomials of complements of those trees described as paths, stars, forks, double forks, and extended double forks. Any tree complement for $n \leq 7$ fits at least one of those descriptions. All nonisomorphic fork complements are proven to be chromatically distinct, as are all nonisomorphic double fork complements with the same handle length.

Closed formulas for the chromatic polynomials of paths P_n, cycles C_n, wheels W_n, and their complements are expressed relative to various bases, and their relationships illustrated. If $n > 3$ is a prime, then n divides b_i, $i = 3,\ldots,n-1$, where b_i is the coefficient of $(\lambda)_i$ in $P(C_n;\lambda)$. For $k \geq 2$, we prove $P(\bar{P}_{2k+1};\lambda) = P(\overline{P_k \cup C_{k+1}};\lambda)$. If G is the graph on n vertices with k disjoint edges and $n - 2k$ isolated vertices, then $\bar{G}$ is chromatically unique.

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CHAPTER 1: INTRODUCTION.

1.1. History and Concepts.

In 1912, George Birkhoff [4] introduced chromatic polynomials in an unsuccessful attempt to solve the four-color conjecture. The proof of the four-color theorem was obtained in 1976 [1, 2] by means of a computer analysis of almost 2000 cases. This proof did not use the theory of chromatic polynomials.

Meanwhile, new methods in chromatic polynomial theory have led to research in directions other than attempting to prove the four-color theorem. The study of chromatic polynomials is currently an active branch of graph theory, with many fundamental problems as yet unsolved. One such problem is to find a necessary and sufficient condition for a polynomial to be the chromatic polynomial of a graph. Another is the classification of all graphs which are uniquely determined, up to isomorphism, by their chromatic polynomial.

For the purposes of this report, a *graph* G is a pair of finite sets (V,E), where V is a non-empty set of n *vertices*, and E is a set of e distinct unordered pairs (a,b), with $a,b \in V$ and $a \neq b$. These pairs are called *edges*. A graph is *planar* if it can be drawn on a plane such that its edges do not cross.

Given a graph G, we can label its vertices 1,...,n. Now we introduce a set of λ colors, and assign a color to each of the n vertices so that two vertices joined by an edge do not receive the same color. Such an assignment is a proper coloring of G; by a coloring of G, we shall mean a proper coloring. Note that not all of the λ colors need be used. If such a coloring is possible, then G is λ-colorable. The smallest integer for which G is λ-colorable is the chromatic number of G, and is designated by $\chi(G)$.

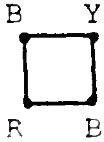
This coloring procedure is a direct abstraction of the cartographer's problem of coloring a map of political regions so that two regions with a common border are colored with distinct colors. Simply think of the regions as vertices; join two vertices by an edge if the corresponding regions have a border in common. Any map the cartographer draws will correspond to a planar graph. The four-color theorem states that all planar graphs are 4-colorable.

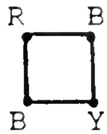
Chromatic polynomials were invented to count the number of different colorings of a graph. What does it mean for two colorings to be different? If we start with a coloring of G, and permute the colors, then we obtain a new coloring, provided the permutation is not the identity. These two colorings are considered to be different; this property is known as color difference. For example, given the set of colors {R,Y,B} representing red, yellow and blue

respectively, the coloring 

is different from 

.

Here the permutation is the one that interchanges the two colors red and yellow. Two colorings are considered to be the same under <u>color indifference</u> if one coloring can be obtained from the other by a permutation of the colors. Under both color difference and color indifference, the vertices of the graph are fixed; i.e., permutations of the vertices are not allowed. Thus 

is different from 

under both color difference and color indifference. From this point on, a coloring will mean a <u>proper coloring with color difference</u>.

In his 1912 paper, Birkhoff showed that the function which describes the number of ways of coloring G, with λ or fewer colors, is in fact a polynomial in λ. The proof is as follows:

Start with a graph G. Let m_i be the number of ways of coloring G by using exactly i colors, with color indifference. There are λ ways of selecting the first of the i colors, $\lambda - 1$ ways of selecting the second of the i colors, $\lambda - 2$ ways of selecting the third color,..., and $\lambda - i + 1$ ways of selecting the i-th color. Thus, the expression $m_i\lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - i + 1)$ represents the number of ways of coloring G by using exactly i colors with color difference.

If i is the number of colors actually used to color G, then i can range from 1 to n, the number of vertices. Summing over i gives the total number of ways G can be colored in λ or fewer colors.

$$P(G;\lambda) = \sum_{i=1}^{n} m_i \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - i + 1)$$

Q.E.D.

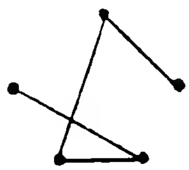
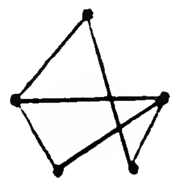
Therefore $P(G;\lambda)$ is a polynomial, since it is expressed as a linear combination of polynomials. $P(G;\lambda)$ is called the <u>chromatic polynomial</u> of G in the variable λ. The falling factorial notation $(\lambda)_i \equiv \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - i + 1)$ will be used in this manuscript.

If the chromatic polynomial is expressed as a linear combination of chromatic polynomials of complete graphs, as it was in Birkhoff's demonstration, we say the <u>chromatic polynomial</u> is <u>represented</u> in the <u>complete graph basis</u>, or that the <u>coefficients</u> of the chromatic polynomial are <u>relative</u> to the complete graph basis. As we shall see in Chapter 2, other representations are possible; the advantages of each are examined and compared.

1.2. Definitions of and Notations for Special Families of Graphs.

The notation for special graphs and their definitions is varied in the literature, so that a short list will serve as a reference. All graphs described below have n vertices and e edges. The *degree* of a vertex is the number of edges incident with it.

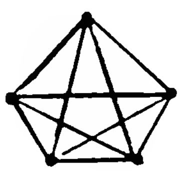
The *complement* of graph G, denoted by $\bar{G}$ (and alternately by G^c if G is described pictorally), has the same vertices as G, but 2 vertices are joined by an edge in $\bar{G}$ if and only if they are not joined by an edge in G.

e.g. G:  $\bar{G}$: 

1. In the *complete graph*, K_n, every pair of vertices is joined by an edge. Thus $e = \binom{n}{2} = \frac{1}{2} n(n - 1)$, and every vertex has degree $n - 1$.

2. The *null* graph contains no edges, and can be written as $\bar{K}_n$. Every vertex has degree 0.

The *empty* graph contains no edges and no vertices.

e.g., K_5  $\bar{K}_3$ 

A graph is *connected* if every pair of distinct vertices is joined by a path.

Graphs may be categorized roughly by the number of edges they possess. A graph with few edges, $e = 0(n)$, is a *sparse* graph. A graph G is *dense* if $\binom{n}{2} - e = 0(n)$.

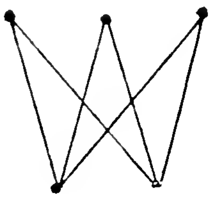
3. If the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1

with a vertex of V_2, we have a <u>bipartite</u> graph. If G contains every possible edge joining V_1 to V_2, G is <u>complete bipartite</u>, and denoted by $K_{i,j}$, where $|V_1| = i$, $|V_2| = j$, $i + j = n$, and $e = i \cdot j$.

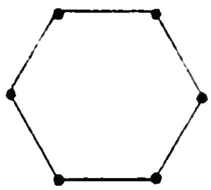
4. The <u>path</u>, P_n, is a graph whose vertices can be labeled such that the edge set consists of $\{(i,i+1) : i = 1,\ldots,n-1\}$. Note $e = n - 1$.

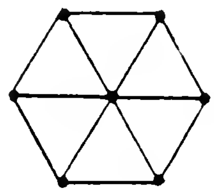
5. The <u>cycle</u>, C_n, is a closed path. It consists of all the edges in the path along with $(n,1)$. Every vertex has degree 2.

6. The <u>wheel</u>, W_n, consists of a cycle on the first $n - 1$ vertices and $\{(i,n) : i = 1,\ldots,n-1\}$

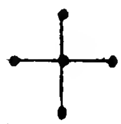

$K_{2,3}$


P_6

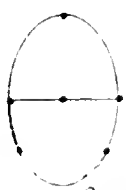

C_6


W_7

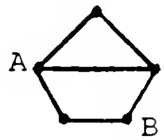
7. A <u>tree</u> on n vertices, T_n, is any connected graph which contains no cycles.

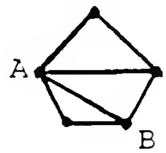
e.g.,    are all the trees on 5 vertices. Note that the path is a special tree. The <u>star</u>, $K_{1,n-1}$, is another special tree.

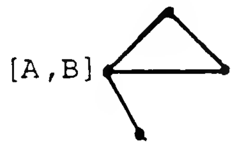
8. A <u>θ-graph</u> is a connected graph consisting of 3 edge-disjoint paths between two vertices of degree 3. All other vertices have degree 2. If these paths have lengths a, b, c respectively with $a \leq b \leq c$, then $\theta_{a,b,c}$ has $n = a + b + c - 1$ vertices and $e = a + b + c$ edges.

   are all the θ-graphs on 6 vertices.

1.3. Some Fundamental Theorems and Properties of Chromatic Polynomials.

Hassler Whitney's 1932 paper [30] contained a fundamental theorem for computing the coefficients of the chromatic polynomial of a graph. In the following example [22, p. 55], consider the graph G: 

, and examine a pair of non-adjacent vertices, say A and B. Two types of colorings of G are possible: those in which A and B are assigned different colors, and those in which A and B are given the same color. Now look at graph G': 

obtained from G by the addition of edge(A,B). In G', A and B <u>must</u> have distinct colors; a coloring of G' is a coloring of G, and conversely, a coloring of the first type of G is a coloring of G'. Next, construct a graph G": 

obtained from G by contracting vertices A and B to one vertex, such that any edge previously joined to A or B is joined to this new vertex. This contraction process is called <u>identifying vertices</u>. Multiple edges are replaced by a single edge, as they both represent the same coloring restriction. Any coloring of G where A and B have the same color will be a coloring of G", and any coloring of G" will be a coloring of G of the second type.

Putting these results together, we see that we can express the chromatic polynomial of a graph G in terms of the chromatic polynomials of a graph G' with an extra edge and a graph G" with one fewer vertices.

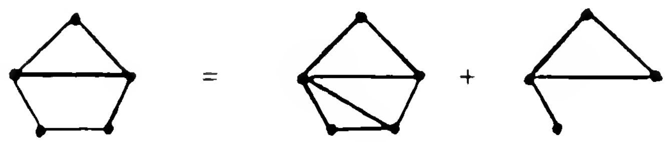
Thus we have proven the following theorem:

<u>Theorem 1.3.1.</u> (Whitney's identity)

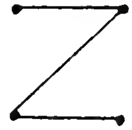
$$P(G;\lambda) = P(G';\lambda) + P(G'';\lambda)$$

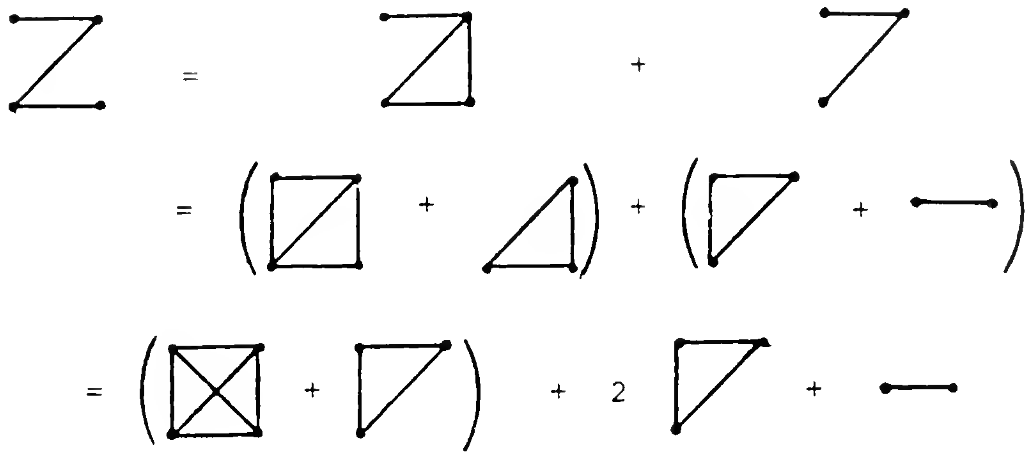
<u>Remark.</u> Although this identity is usually attributed to Whitney, Whitney [30] gives R. M. Forster credit for this result. This method of computing has been very useful. All implemented algorithms known to the author, including the one implemented for this manuscript using the complete graph basis (see Section 2.3), are based on this theorem.

Zykov [33] introduced a clever notational device—we let the actual drawing of the graph denote its chromatic polynomial. Thus, in the above example, instead of writing $P(G;\lambda) = P(G';\lambda) + P(G'';\lambda)$, we write



By repeatedly applying Theorem 1.3.1, we eventually end up with the sum of chromatic polynomials of complete graphs. This process is finite because at each stage, we are adding an edge (at most $\binom{n}{2}$ edges) and deleting a vertex (at most n vertices). It is easy to show that the chromatic polynomial of a complete graph is simply $P(K_n;\lambda) = (\lambda)_n$. Since any graph is the sum of chromatic polynomials of complete graphs, we have another affirmation that the chromatic polynomial is indeed a polynomial. This fact can be proven formally using induction and Theorem 1.3.1.

To complete the illustration, let us compute the chromatic polynomial of the graph H: 

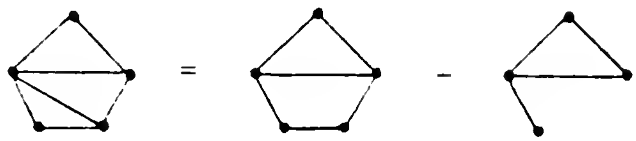


Thus $P(H;\lambda) = (\lambda)_4 + 3(\lambda)_3 + (\lambda)_2$. Should it be necessary, we can easily convert this to the polynomial representation in the null graph basis; namely, $P(H;\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda$, using Stirling numbers. (See Section 2.1.)

If we are computing the chromatic polynomial of a graph with few edges, it is easier to use Corollary 1.3.1 by deleting edges rather than adding them.

Corollary 1.3.1. $P(G';\lambda) = P(G;\lambda) - P(G'';\lambda)$.

Thus, our old example G becomes



Other well-known observations, see Read [22], reduce the amount of work needed to compute chromatic polynomials.

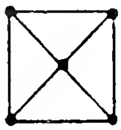
Theorem 1.3.2. If a graph G has connected components $G_1, G_2, \ldots, G_k$ then $P(G;\lambda) = P(G_1;\lambda)P(G_2;\lambda)\cdots P(G_k;\lambda)$.

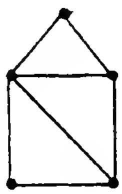
Proof. Disjoint components can be colored independently of each other. Therefore the number of colorings of the whole graph is the product of the number of colorings of the individual components.

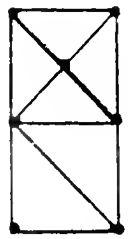
Theorem 1.3.3. (Zykov [33]) If two graphs G and H have complete graph K_r in common (i.e., overlap in K_r), then the chromatic polynomial of the graph formed by G and H together is

$$\frac{P(G;\lambda)P(H;\lambda)}{(\lambda)_r}$$

e.g.

G: 

H: 

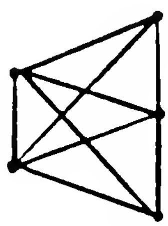
The graph formed by G and H together (overlapping in K_3): 

Proof. The number of colorings of the common K_r is $(\lambda)_r$. Fix the colors of these r vertices. Then there are $\frac{P(G;\lambda)}{(\lambda)_r}$ ways of coloring the remaining vertices of G, and $\frac{P(H;\lambda)}{(\lambda)_r}$ ways of coloring the remaining vertices of H. Hence the total number of colorings is $(\lambda)_r \cdot \frac{P(G;\lambda)}{(\lambda)_r} \cdot \frac{P(H;\lambda)}{(\lambda)_r} = \frac{P(G;\lambda)P(H;\lambda)}{(\lambda)_r}$.

In 1949, Zykov [33] defined the product of 2 disjoint graphs, G ⊙ H, as the graph obtained by adding all possible edges joining a vertex in G to a vertex in H. For example:

G: 

H: 

G ⊙ H: 

Theorem 1.3.4. (Zykov [33]) Given 2 disjoint graphs G and H with chromatic polynomials $P(G;\lambda)$ and $P(H;\lambda)$ represented in the complete graph basis. Then the chromatic polynomial of $G \odot H$ is $P(G \odot H;\lambda) = P(G;\lambda) \odot P(H;\lambda)$, where the polynomial operator $\odot$ denotes the operation in which factorials are multiplied as powers. This operation is known as umbral multiplication of polynomials and is found in Riordan's book [25].

Proof. If G has m vertices and H has n vertices then,

$$P(G;\lambda) = a_m(\lambda)_m + a_{m-1}(\lambda)_{m-1} + a_{m-2}(\lambda)_{m-2} + \cdots ,$$

$$P(H;\lambda) = b_n(\lambda)_n + b_{n-1}(\lambda)_{n-1} + b_{n-2}(\lambda)_{n-2} + \cdots .$$

If we now use Whitney's identity by adding edges to $G \odot H$, the G and H portions of the product graph will be exactly as above. At each use of the identity, every vertex of every graph obtained from decomposing G will be joined to every vertex of each graph decomposed from H. At the end we have all possible products of complete graphs that are summed in $P(G;\lambda)$ and those summed in $P(H;\lambda)$. But the product of a complete graph on p vertices and a complete graph on q vertices is a complete graph on $(p + q)$ vertices. Therefore a term $(\lambda)_p$ in $P(G;\lambda)$ and a term $(\lambda)_q$ in $P(H;\lambda)$ will give rise to a term $(\lambda)_{p+q}$ in $P(G \odot H;\lambda)$, i.e. factorials multiply like powers.

Q.E.D.

For our example, $P(G;\lambda) = (\lambda)_2$, $P(H;\lambda) = (\lambda)_3 + (\lambda)_2$; therefore $P(G \odot H;\lambda) = (\lambda)_2 \odot ((\lambda)_3 + (\lambda)_2) = (\lambda)_5 + (\lambda)_4$.

Some properties of chromatic polynomials that will be useful later, are briefly stated here. The graph G has n vertices.

Property 1.3.1. The degree of $P(G;\lambda)$ is n.

By the Whitney identity reduction method, there is exactly 1 graph having n vertices at each stage, including the final one. From this immediately follows the next property.

Property 1.3.2. The coefficient in $P(G;\lambda)$ of the term of degree n is 1.

Property 1.3.3. $P(G;\lambda)$ has no constant term.

If $P(G;\lambda)$ had a constant term, say k, then $P(G;0) = k$. This means that graph G can be colored in k ways using zero colors! Therefore, k must be zero.

Interpreting the other coefficients of $P(G;\lambda)$ will take place in Section 2.2, when we examine representations of the chromatic polynomial in different bases.

2.1. Representation of a Chromatic Polynomial in the Null Graph, Tree, and Complete Graph Bases.

The set of polynomial functions over the field of rational numbers is a vector space, with the usual operations of polynomial addition and multiplication. Chromatic polynomials are contained in the subspace whose vectors are of finite length, and where the components of the vector are integers. Chromatic polynomials are not a subspace by themselves because scalar multiplication of a chromatic polynomial by a number other than 1 produces a polynomial which is not a chromatic polynomial.

A set of polynomials S spans the set of chromatic polynomials if any chromatic polynomial can be written as a linear combination of elements of S. This set S is linearly independent if no element in S can be written as a linear combination of other elements in S. A basis for the set of chromatic polynomials is a linearly independent set of polynomials spanning the set of chromatic polynomials. When we write a chromatic polynomial, we are actually listing the coefficients of this polynomial relative to a given basis. The three most commonly used bases are given below:

I. A chromatic polynomial for graph G expressed in the null graph basis (also called the standard basis) is written as a unique, finite, linear combination of $\{1, \lambda, \lambda^2, \ldots, \lambda^n, \ldots\}$; i.e., for graph G with n vertices,

$$P(G;\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + (-1)^{n-1}a_1\lambda \ ,$$

$$a_i \geq 0 \ , \quad i = 1,\ldots,n-1 \ .$$

As explained in Section 1.3, the coefficient of λ^n is always 1, and $P(G;\lambda)$ has no constant term. The "1" of this and all other bases can be interpreted as the chromatic polynomial of the <u>empty</u> graph—the graph having no vertices and no edges. This basis is called the null graph basis because $P(\bar{K}_n;\lambda) = \lambda^n$; each of the n vertices of the null graph can be colored independently in λ ways. A well-known result enables us to represent the coefficients as they are above.

<u>Theorem 2.1.1</u>. For any graph G, the coefficients of its chromatic polynomial relative to the null graph basis alternate in sign.

<u>Proof</u>. The proof proceeds by 2-way induction. For n = 1, 2, and 3, the following graphs verify the theorem:

	λ		$\lambda^3 - \lambda^2$
	λ^2		$\lambda^3 - 2\lambda^2 + \lambda$
	$\lambda^2 - \lambda$		$\lambda^3 - 3\lambda^2 + 2\lambda$
	λ^3		

Let the theorem be true for all graphs with n vertices or less. Consider the graphs with n + 1 vertices.

For the null graph $\bar{K}_{n+1}$, $P(\bar{K}_{n+1};\lambda) = \lambda^{n+1}$, and the theorem is true. Now let the theorem be true for all graphs with n + 1 vertices and k edges or less. For any graph G' with n + 1 vertices and k + 1 edges, Corollary 1.3.1 states that $P(G';\lambda) = P(G;\lambda) - P(G'';\lambda)$, where G has n + 1 vertices and k edges, and G" has n vertices and $\leq$ k edges. Since the theorem is true for G and G",

$$P(G;\lambda) = \lambda^{n+1} - a_{1,n}\lambda^{n} + a_{1,n-1}\lambda^{n-1} + \cdots + (-1)^{n}a_{1,1}\lambda ,$$

$$a_{1,i} \geq 0 , \qquad i = 1,n ,$$

$$P(G'';\lambda) = \lambda^{n} - a_{2,n-1}\lambda^{n-1} + \cdots + (-1)^{n-1}a_{2,1}\lambda ,$$

$$a_{2,i} \geq 0 , \qquad i = 1,n - 1 .$$

Thus $P(G';\lambda) = \lambda^{n+1} - (a_{1,n} + 1)\lambda^{n} + (a_{1,n-1} + a_{2,n-1})\lambda^{n-1} + \cdots + (-1)^{n}(a_{1,1} + a_{2,1})\lambda$, where all the coefficients alternate in sign. Q.E.D.

II. A chromatic polynomial for graph G expressed in the <u>tree basis</u> is written as a unique, finite, linear combination of $\{1,\lambda,\lambda(\lambda - 1),\lambda(\lambda - 1)^{2},\ldots,\lambda(\lambda - 1)^{n-1},\ldots\}$; i.e. $P(G;\lambda) = \lambda(\lambda - 1)^{n-1} + c_{n-1}\lambda(\lambda - 1)^{n-2} + \cdots + c_{1}\lambda$, c_i integer, $i = 1,\ldots,n - 1$. This is called the tree basis because of the following:

<u>Theorem 2.1.2.</u> Any tree T_n with n vertices has chromatic polynomial $P(T_n;\lambda) = \lambda(\lambda - 1)^{n-1}$.

<u>Proof</u>. By induction. For n = 1, 2, and 3, we saw in the previous proof that

 $\lambda = \lambda(\lambda - 1)^{0}$

 $\lambda^{2} - \lambda = \lambda(\lambda - 1)^{1}$

 $\lambda^{3} - 2\lambda^{2} + \lambda = \lambda(\lambda - 1)^{2}$

Let the theorem be true for all trees on n vertices. For any tree on n + 1 vertices, choose a <u>terminal vertex</u>, a vertex with degree 1. Any tree on 2 or more vertices always has a terminal vertex.

Now use Whitney's identity (Corollary 1.3.1):

$$\begin{aligned} P(T_{n+1};\lambda) &= P(T_1;\lambda)\, P(T_n;\lambda) - P(T_n;\lambda) \\ &= \lambda(\lambda(\lambda - 1)^{n-1}) - \lambda(\lambda - 1)^{n-1} \\ &= \lambda(\lambda - 1)^{n-1}(\lambda - 1) = \lambda(\lambda - 1)^n . \end{aligned}$$

Q.E.D.

Theorem 2.1.3. For any connected graph G, the coefficients of its chromatic polynomial relative to the tree basis alternate in sign.

Proof. Nijenhuis and Wilf [21] calculate chromatic polynomials by deleting edges with Corollary 1.3.1 until we can represent the chromatic polynomial of G as the sums and differences of chromatic polynomials of trees. At each stage an edge is chosen so that its removal will not disconnect the graph. This can always be done because a connected graph contains a spanning tree as a subgraph. Hence, if c_j is the number of trees on j vertices which are produced by the above algorithm, then $P(G;\lambda) = \sum_{j=1}^{n} (-1)^{n-j} c_j \lambda(\lambda - 1)^{j-1}$, $c_j \geq 0$, $j = 1,\ldots,n$. In particular, $c_n = 1$, as we will see in Section 2.2.

Connectedness plays a crucial role in tree basis representations. A graph that is not connected does not have a spanning tree. At some point in the algorithm, we will have to add at least 1 edge to obtain a tree, rather than continue the process of deleting edges. Adding an edge destroys the alternating sign property.

III. A chromatic polynomial for graph G represented in the <u>complete graph basis</u> (also called the falling factorial basis) is written as a unique, finite, linear combination of $\{1, (\lambda)_1, (\lambda)_2, \ldots, (\lambda)_n, \ldots\}$; i.e., if G has n vertices,

$$P(G;\lambda) = (\lambda)_n + b_{n-1}(\lambda)_{n-1} + b_{n-2}(\lambda)_{n-2} + \cdots + b_1(\lambda)_1 , \quad b_i \geq 0 , \quad i = 1,\ldots,n-1 .$$

As we saw previously, $P(K_n;\lambda) = (\lambda)_n$. All the coefficients relative to the complete graph basis are non-negative. The proof proceeds by noting that the end product of repeated applications of Whitney's identity (Theorem 1.3.1) yields a sum of complete graphs.

For any two bases, there is always a linear transformation which expresses the coordinates (the coefficients of the chromatic polynomial) relative to one basis in terms of the other [13]. The matrix representation for this transformation between the null graph, tree, and complete graph bases is always a triangular matrix, and can be stored using $\frac{n^2 + n}{2}$ locations, where n is the maximum number of vertices under analysis. For the null and complete graph bases we have:

$$(\lambda)_n = \sum_{k=0}^{n} s(n,k)\lambda^k , \quad \text{with} \quad s(0,0) = 1 ;$$

$$s(n,0) = 0 , \qquad n > 0 ,$$

$$s(n+1,k) = s(n,k-1) - ns(n,k) , \quad 0 < k \leq n ,$$

and

$$\lambda^n = \sum_{k=0}^{n} S(n,k)(\lambda)_k , \quad \text{with} \quad S(0,0) = 1 ;$$

$$S(n,0) = 0 , \qquad n > 0 ,$$

$$S(n+1,k) = S(n,k-1) + kS(n,k) , \quad 0 < k \leq n ,$$

where $s(n,k)$ are the Stirling numbers of the first kind and $S(n,k)$ are the Stirling numbers of the second kind. For tables of these Stirling numbers, see Appendix A of [19A].

Chao and Whitehead [5,29] have shown that

$$\lambda(\lambda - 1)^{n-1} = \sum_{k=0}^{n} t(n,k)\lambda^k , \quad n \geq 1 ,$$

$$\text{where} \quad t(n,k) = (-1)^{n+k}\binom{n-1}{k-1} ,$$

and

$$\lambda^n = \sum_{k=0}^{n} T(n,k)\lambda(\lambda - 1)^{k-1} , \qquad n \geq 1 ,$$

$$\text{where } T(n,k) = \left\{ {n-1 \atop k-1} \right\}, \text{ such that}$$

$$\sum_{j=1}^{i} T(i,j)t(j,k) = \sum_{j=1}^{i} t(i,j)T(j,k) = \delta_{ik} ,$$

i.e., the matrix $[T(n,k)]$ is the inverse matrix of $[t(n,k)]$. From Figure 2.1.1, we see that $[T(n,k)] = [S(n,k)]\cdot[1 \oplus s(n,k)]$, where $[1 \oplus S(n,k)]$ and $[1 \oplus s(n,k)]$ are defined as

$$\left[\begin{array}{c|c} 1 & \cdots\cdots \\ \hline \vdots & S(n,k) \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} 1 & 0 \cdots\cdots \\ \hline 0 & \\ \vdots & s(n,k) \end{array}\right]$$

Then Chao and Whitehead [5,29] also showed

$$(\lambda)_n = \sum_{k=1}^{n} s(n-1,k-1)\,\lambda(\lambda - 1)^{k-1} ,$$

and

$$\lambda(\lambda - 1)^n = \sum_{k=1}^{n} S(n-1,k-1)\,(\lambda)_k .$$

Figure 2.1.1 summarizes these transformations.

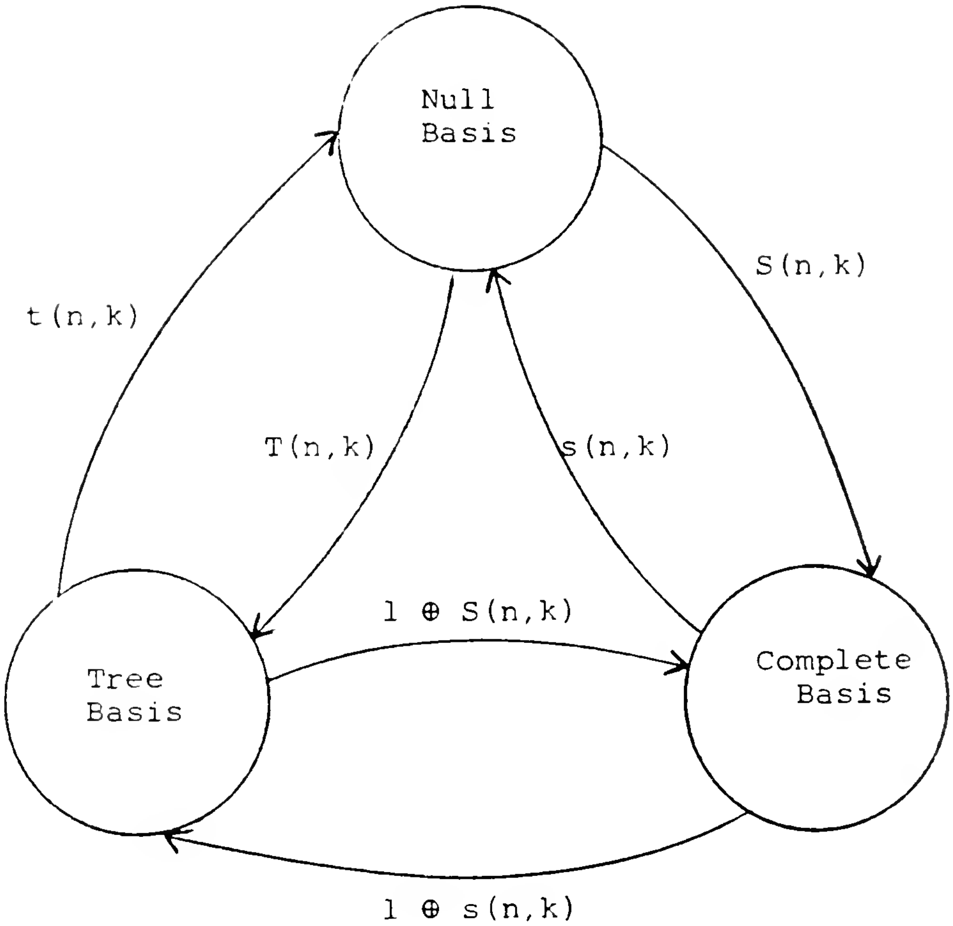

Null Basis
S(n,k)
t(n,k)
T(n,k)
s(n,k)
1 ⊕ S(n,k)
Tree Basis
Complete Basis
1 ⊕ s(n,k)

FIGURE 2.1.1

2.2. Relative Merits of the Null Graph, Tree, and Complete Graph Bases, with Interpretation of Coefficients.

Each of the three bases has advantages, either in computing the chromatic polynomial for a particular graph or in analysis of its coefficients.

The null graph basis is useful for computation of the chromatic polynomial whenever G is sparse and disconnected. We compute the chromatic polynomial of each connected component, and multiply the polynomials using the usual polynomial arithmetic. It is not as convenient to multiply polynomials in either the tree or complete graph bases. The easiest way to do it would be to use matrix transformations of Section 2.1 to convert to the null graph basis first! If a graph is sparse and connected, the tree basis computation will always require the deletion of less edges than the null graph basis, since we always arrive at a tree before we arrive at the null graph. Thus, less work is involved. When the graph is dense it is easier to add edges and terminate with complete graphs. For these, the complete graph basis is best.

As was shown in Section 2.1, the coefficients of the chromatic polynomial relative to the null graph basis alternate in sign for all graphs. Relative to the tree basis, they alternate in sign for all connected graphs, and relative to the complete graph basis, the sign is always positive.

In addition to having only non-negative coefficients, the complete graph basis affords easy use of Theorem 1.3.3, computation of the chromatic polynomial of a graph consisting of 2 subgraphs overlapping in K_r, and Theorem 1.3.4, the computation of the chromatic polynomial of a graph that can be written as the Zykov product of 2 graphs. The coeffi-

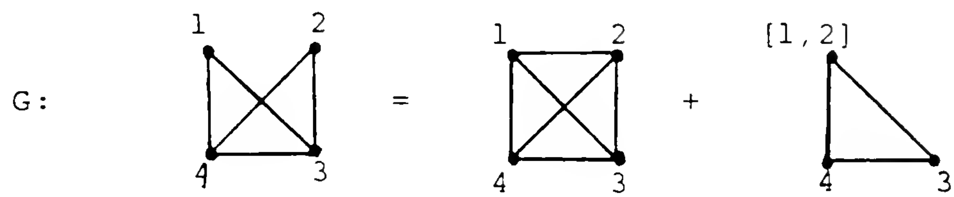
cients represented in the complete graph basis are easily interpreted. If $P(G;\lambda) = \sum_{i=0}^{n} b_i(\lambda)_i$, with $b_n = 1$, $b_i \geq 0$, then b_i is the number of ways of coloring G using exactly i colors with color indifference. This is a direct outcome of Birkhoff's original proof in Section 1.1.

The following are well-known properties, whose proofs can be found in Read [22].

<u>Property 2.2.1</u>. The smallest number k for which $b_k > 0$ is the chromatic number of G, i.e. $\chi(G) = k$.

In addition to generating the chromatic number, the process of computing the chromatic polynomial via Theorem 1.3.1 yields an actual coloring of the graph G in $\chi(G)$ colors. By keeping track of vertex labels, the graphs $K_{\chi(G)}$ will be labeled with sets of labels from the original graph. All the labels in one set are colored the same color.

As an illustration, consider



Thus $\chi(G) = 3$; looking at K_3, the vertices are labeled [1,2], 3, and 4 respectively. Thus, a possible 3-coloring of the original graph G colors vertices 1 and 2 using the 1st color, vertex 3 using the 2nd color, and vertex 4 using the 3rd color.

The coefficients of the chromatic polynomial relative to the null graph basis are not as easily interpreted, although some results are clear. If $P(G;\lambda) = \sum_{i=0}^{n} (-1)^{n-i} a_i \lambda^i$, with $a_n = 1$, then the following are true.

<u>Property 2.2.2</u>. $a_{n-1} = e$, the number of edges in G.

<u>Property 2.2.3</u>. If G is connected, $a_r \geq \binom{n-1}{r-1}$.

<u>Property 2.2.4</u>. The smallest number k for which $a_k > 0$ is the number of components of G.

<u>Theorem 2.2.1</u>. G is a tree if and only if $P(G;\lambda) = \lambda(\lambda - 1)^{n-1}$.

<u>Proof</u>. If G is a tree, $P(G;\lambda) = \lambda(\lambda - 1)^{n-1}$, as per Section 2.1. If $P(G;\lambda) = \lambda(\lambda - 1)^{n-1}$, then G is connected ($a_1 \neq 0$), and has $n - 1$ edges ($a_{n-1} = n - 1$), by Properties 2.2.4 and 2.2.2 respectively. A connected graph with n vertices and $n - 1$ edges is a tree.

<u>Corollary 2.2.1</u>. (Eisenberg [8]) G is a tree if and only if $a_1 = 1$.

<u>Theorem 2.2.2</u>. $a_r = \sum_{i=0}^{e} (-1)^{n-r+i} N(r,i)$, where $N(r,i)$ is the number of subgraphs of G with r components and i edges.

This interpretation of the coefficients of the chromatic polynomial is difficult to use in computation.

Whitney [31] showed that it is enough to consider $N(r,i)$ as the subgraphs of G which do not contain any broken cycles. This broken cycle theorem is more useful in computation (see proofs in Section 3.1) but not often used.

Read's results can be extended to other bases.

Property 2.2.5. Let $P(G;\lambda) = \sum_{i=1}^{n} c_i \lambda(\lambda - 1)^{i-1}$.

If G is connected, $|c_{n-1}| = e - (n - 1)$.

Proof. This can be seen by observing that in each step of computing the chromatic polynomial of G via Corollary 1.3.1, every deletion of an edge adds 1 to c_{n-1}. We stop when we arrive at the spanning tree by G, which has n - 1 edges.

Property 2.2.6. Eisenberg [8]. Let $P(G;\lambda) = \sum_{i=1}^{n} b_i (\lambda)_i$. Then $b_{n-1} = \binom{n}{2} - e$.

These three bases have one property in common. Starting with a connected graph, and using Whitney's identity, we either only delete edges or only add edges to arrive at the basis elements. Adding and deleting edges are not combined. No other family of graphs known to the author form a basis which preserves this property. Therefore, chromatic polynomials are most easily computed in the null, tree, and complete graph bases.

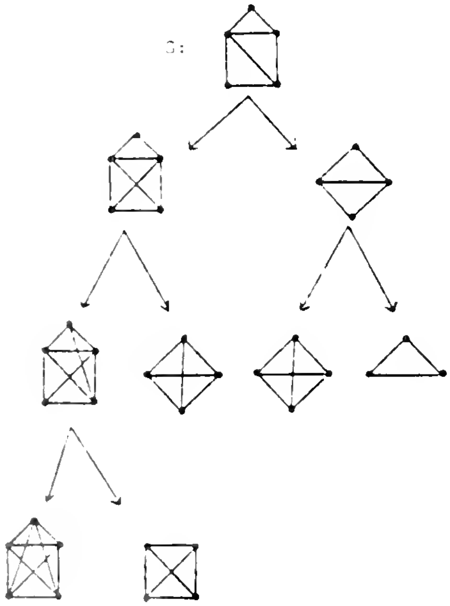
2.3. An Algorithm for Computing Chromatic Polynomials in the Complete Graph Basis.

Karp [15] showed that computing the chromatic number of a graph is an NP-complete problem. Therefore, the best known algorithms for this computation are exponential in the input, in worst case behavior. For this problem, the input size is $O(n^2)$, where n is the number of vertices of graph G.

Computing the chromatic polynomial of a graph is at least as difficult as computing the chromatic number, and in fact the chromatic polynomial contains the chromatic number as the smallest subscript of the nonzero coefficients of the chromatic polynomial expressed relative to the complete graph basis. Thus, the calculation of the chromatic polynomial is NP-hard.

The algorithm described here is based on Whitney's identity. We would like to compute the chromatic polynomial of graph G by adding edges and identifying vertices until we arrive at a sum of chromatic polynomials of complete graphs equal to the chromatic polynomial of G. Algorithms already in the literature delete edges, either to express the chromatic polynomial using the null graph basis [24], or in the case of connected graphs, to express the chromatic polynomial using the tree basis [21]. They are also based on Whitney's identity. For a dense graph, there are less edges to add to arrive at a complete graph than edges to delete to arrive at a tree; as such, the sum of the absolute values of the coefficients for this graph will be smaller, in general, relative to the complete graph basis than the tree basis.

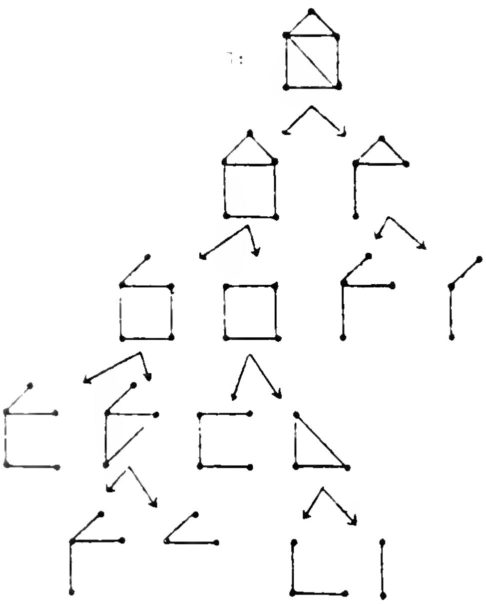
To see this more clearly, consider the binary tree below generated by the computation of the chromatic polynomial of G.



At each step, the left-most branch represents the addition of an edge, while the right-most branch represents the identification of the corresponding vertices.

$$P(G;\lambda) = (\lambda)_5 + 3(\lambda)_4 + (\lambda)_3.$$

If we were to delete edges by using the tree basis, we would find the following binary tree generated:



At each step the left-most branch represents the deletion of an edge, while the right-most branch represents the identification of the corresponding vertices.

$$P(G;\lambda) = \tau_5 - 3\tau_4 + 3\tau_3 - \tau_2$$

where $\tau_j = P(T_j;\lambda) = \lambda(\lambda-1)^{j-1}$

When using the null or complete graph basis, or tree basis for connected graphs, each terminal vertex of this binary tree adds 1 to one of the coefficients of $P(G;\lambda)$. Let $S(G)$ denote the sum of the absolute values of the coefficients of $P(G;\lambda)$. Then $S(G)$ is the number of terminal vertices of this binary tree. In our example, $S(G) = 5$ for the complete graph basis representation and $S(G) = 8$ for the tree basis representation.

By Knuth [16], if a binary tree has i terminal vertices, or leaves, then it has i - 2 nonterminal, or internal vertices, and a special vertex called the root, where the original graph G is placed.

Proof. Consider any binary tree with n vertices, n - 1 edges, and total degree $2(n - 1)$.

This tree has one root of degree 2, i leaves of degree 1, and $n - i - 1$ internal vertices of degree 3.

$$2\cdot 1 + i + 3(n - i - 1) = 2(n - 1)$$

Rearranging terms yields

$$n-i-1 = i-2 \qquad \text{Q.E.D.}$$

This means that in the process of calculating $P(G;\lambda)$ we are generating $2S(G) - 2$ other graphs. Therefore, by suitable algorithm basis choice, we can minimize $S(G)$, thus reducing the computation required to arrive at $P(G;\lambda)$.

Once the chromatic polynomial of G is obtained, in whatever basis, its coefficients can be transformed to those of another basis via the matrices of Section 2.1. Since changing bases is only a polynomial bounded problem, it is of least concern when calculating $P(G;\lambda)$.

This new algorithm CCHROM, is designed to be storage-efficient for dense graphs G by storing and working with their complements. Input to CCHROM is graph G, described by the number of vertices NODES, the number of edges NOEDGE, and edge list NPTS, where NPTS(1,K) = I, NPTS(2,K) = J means the K-th edge joins vertices I and J, K = 1,...,NOEDGE, with I < J. The output is 3 arrays, NN, NT, NC, containing the coefficients of $P(G;\lambda)$ in the null graph, tree, and complete graph bases, respectively.

Graphs in CCHROM are stored using the edge list of their complements, created by subroutine CMPLMT via an asymmetric NODES $\times$ NODES adjacency matrix ING. ING will also be used to keep track of edges which are involved in the vertex identification process. A stack of pairs, ISTACK, will store the edge list of the complement of graphs waiting to be analyzed; NSTACK contains the number of graphs on ISTACK. NBEGIN(K) and NEND(K) hold the beginning and end positions of the K-th graph on ISTACK, while NONODE(K) tells the number of vertices in the K-th graph.

<u>ALGORITHM CCHROM</u>:

[1] Initialization. NC(I) $\leftarrow$ 0, I = 1,...,NODES.

if NOEDGE = $\binom{\text{NODES}}{2}$, NC(NODES) = 1. GO TO [4];

else ISTACK $\leftarrow$ $\overline{G}$, NSTACK $\leftarrow$ NSTACK + 1.

[2] If NSTACK = 0, GO TO [4].

Get the next graph $\bar{G}$, with NV vertices and NE edges, from ISTACK; NSTACK $\leftarrow$ NSTACK - 1; perform Whitney's identity and obtain $\overline{GA}$, complement graph with edge deleted [NE $\leftarrow$ NE - 1] and $\overline{GI}$, complement graph with vertices identified, with KV $\leftarrow$ NV - 1 vertices and KE < NE edges.

[3] If NE = 0, NC(NV) ← NC(NV) + 1; else ISTACK ← $\overline{GA}$, NSTACK ← NSTACK + 1; if KE = 0, NC(NV) ← NC(KV) + 1; else ISTACK ← $\overline{GI}$, NSTACK ← NSTACK + 1; GO TO [2].

[4] $P(G;\lambda) = \sum_{I=1}^{NODES} NC(I)\ (\lambda)_I$. Convert to obtain representations in null graph and tree bases. EXIT.

There are at most NODES graphs on ISTACK at any time, each with at most $\binom{NODES}{2}$-NOEDGE edges. These graphs have the property that their vertex and edge counts are strictly decreasing, so that the maximum number of edges on ISTACK is

$$\sum_{I=0}^{NODES} \binom{NODES}{2} - NOEDGE - I = \binom{NODES}{2}(NODES - 1) - NODES*NOEDGE.$$

If $NCEDGE = \binom{NODES}{2}$-NOEDGE,

an upper bound for S(G) is 2^{NCEDGE}, since there are at most NCEDGE levels in the binary tree of graphs generated by CCHROM. Each of these intermediate graphs require 0(NCEDGE) computation, so that this is a $0(NCEDGE \cdot 2^{NCEDGE})$ algorithm.

Another method of traversing the binary tree of graphs generated by a Whitney's identity algorithm is by root, left subtree, right subtree, i.e., preorder (see Knuth [16]). Examine the first computation of the chromatic polynomial of G in this section.

Note that, starting at the top, the leftmost branch terminates in a complete graph of 5 vertices. In general, starting with a graph on N vertices, the leftmost branch terminates in a complete graph of N vertices. This can be described by CCHROM2, which differs from CCHROM in steps [2] and [3].

ALGORITHM CCHROM2:

[1] Initialization. $NC(I) \leftarrow 0$, $I = 1$, NODES.

if $NOEDGE = \binom{NODES}{2}$, $NC(NODES) = 1$. GO TO [4];

else ISTACK $\leftarrow \bar{G}$, NSTACK $\leftarrow$ NSTACK + 1.

[2] If NSTACK = 0, go to [4].

Get the next graph $\bar{G}$, with NV vertices and NE edges, from ISTACK; NSTACK $\leftarrow$ NSTACK - 1.

[3] Process each edge of $\bar{G}$ in turn by creating two graphs:

(a) graph with edge deleted, $\overline{GA}$. This is equivalent to adding the corresponding edge in the original graph. $\overline{GA}$ does not have to be stacked because we will continue deleting complement edges until we are left with a null complement (complete original graph). This essentially represents the leftmost path of any subtree in the computation process.

(b) graph with edge contracted, $\overline{GI}$. This graph has 1 less vertex than graph (a). If it is null (i.e. the original graph is complete), then $NC(NV - 1) \leftarrow NC(NV - 1) + 1$ and continue [3]; else, ISTACK $\leftarrow \overline{GI}$; NSTACK $\leftarrow$ NSTACK + 1.

Upon completion of [3], GO TO [2].

[4] $P(G;\lambda) = \sum_{I=1}^{NODES} NC(I)\ (\lambda)_I$. Convert to obtain representations in null graph and tree bases. EXIT.

CCHROM2 requires a much larger stack than CCHROM, since there can be as many as $\binom{NODES}{2}$-NOEDGE graphs on the stack at once, each with as many as (NODES - 1) vertices. However, we have the advantage of knowing which generated graphs are complete, and as such, do less stack additions and deletions. The result is better runtime.

The code for these algorithms, implemented in CDC FORTRAN, can be found in [19A, Appendices E, F].

2.4. Comparative Computational Efficiency of the Null Graph, Tree, and Complete Graph Bases.

Algorithm CCHROM2 of Section 2.3 was used to generate the chromatic polynomial of all 1044 graphs on seven vertices relative to the complete graph basis; these polynomials were then represented in the null graph and tree bases via basis transformations. Edge lists for these graphs were obtained from Read [24]. In order to establish the amount of computation that would have been required to arrive directly at the chromatic polynomial via the null graph basis and tree basis algorithms, we computed $S(G)$, the sum of the absolute values of the coefficients of $P(G;\lambda)$, for each basis representation. Table 2.4.1 summarizes the results for all graphs on seven vertices, and for all connected graphs on seven vertices.

In computing chromatic polynomials of disconnected graphs, it is best to compute the chromatic polynomial of each connected component, and then multiply the component chromatic polynomials together, by Theorem 1.3.2. Since computing chromatic polynomials is NP-hard, and multiplying polynomials is a polynomial bounded problem, this connected component procedure is optimal.

In view of the above remark, and bearing in mind that the tree basis algorithm works only for connected graphs, we can focus our attention on Table 2.4.1 for connected graphs. From the data, it is clear that an algorithm which

TABLE 2.4.1

ALL GRAPHS WITH 7 VERTICES

NO.EDGES	NO.GRAPHS	............SUM OF ABS(COEFFICIENTS)........ NULL BASIS	TREE BASIS	COMPLETE BASIS
0	1	1	64	477
1	1	2	32	674
2	2	[illegible]	32	1046
3	5	3[illegible]	44	2050
4	10	150	48	3170
5	21	600	66	5209
6	41	2136	77	7758
7	65	5600	248	9668
8	97	14516	639	11535
9	131	30848	1622	12111
10	148	52264	3268	10584
11	148	75262	5481	8112
12	131	92716	7711	5477
13	97	92790	8600	3051
14	65	81684	8314	1526
15	41	65610	7216	[illegible]
16	21	42542	4916	253
17	10	24984	3072	82
18	5	15216	1952	28
19	2	7320	984	7
20	1	4320	600	2
21	1	5040	720	1
TOTAL	1044	613567	55742	42[illegible]
AVG/GRAPH		587.70	53.39	40.74

ALL CONNECTED GRAPHS WITH 7 VERTICES

NO.EDGES	NO.GRAPHS	............SUM OF ABS(COEFFICIENTS)........ NULL BASIS	TREE BASIS	COMPLETE BASIS
6	11	754	11	2233
7	33	3456	95	5182
8	67	11158	409	8198
9	107	25892	1310	10079
10	132	45718	2959	9508
11	138	72238	5195	7674
12	126	93754	7551	5313
13	95	91856	8488	3015
14	64	81084	8238	1513
15	40	64890	7118	[illegible]
16	21	42542	4918	253
17	10	24984	3072	83
18	5	15216	1952	28
19	2	7320	984	7
20	1	4320	600	2
21	1	5040	720	1
TOTAL	853	[illegible]	[illegible]	[illegible]
AVG/GRAPH		[illegible]	[illegible]	[illegible]

computes chromatic polynomials in the null graph basis is slower by a factor of approximately 11 than either the tree or complete graph basis algorithm.

A hybrid algorithm of both the tree and complete graph algorithms is indicated, as follows:

1. If $e \leq \left\lceil \frac{1}{2} \binom{n}{2} \right\rceil$, compute $P(G;\lambda)$ with the tree basis algorithm.
2. If $e > \left\lceil \frac{1}{2} \binom{n}{2} \right\rceil$, use the complete graph basis algorithm to arrive at $P(G;\lambda)$.

For $n = 7$, $\left\lceil \frac{1}{2} \binom{n}{2} \right\rceil = 11$. The following table lists $S(G)$ for the hybrid algorithm:

All connected graphs on 7 vertices

No. Edges	No. Graphs	Sum of Abs(Coefficients) Hybrid Algorithm	
6	11	11	Tree Basis
7	33	90	
8	67	409	
9	107	1310	
10	132	2950	
11	138	5195	
12	126	5313	Complete Graph Basis
13	95	3015	
14	64	1513	
15	40	688	
16	21	253	
17	10	83	
18	5	28	
19	2	7	
20	1	2	
21	1	1	
TOTAL	853	20868	
AVG/GRAPH		24.46	

This hybrid algorithm requires only 38.96% as much computation as the tree basis algorithm (62.79) and only 38.73% as much computation as the complete graph basis algorithm (63.16), on average.

From a table in Harary and Palmer [11, p. 240] we see that there are 12,346 graphs on 8 vertices and 274,668 graphs on 9 vertices. How many of these graphs are connected is not indicated. Generating chromatic polynomials for these graphs would be one way to gain some insight into properties of chromatic polynomials, and to test the hybrid algorithm turning point $\left\lceil \frac{1}{2} \binom{n}{2} \right\rceil$, the number of edges at which to switch from the tree algorithm to the complete graph algorithm. A better way to test this switching point is by the generation of random graphs on, say 8, vertices, testing for connectivity, and, if connected, computing chromatic polynomials in both bases; calculate $S(G)$ for both bases, and compare. We would be interested in concentrating on random graphs with around $\left\lceil \frac{1}{2} \binom{n}{2} \right\rceil$ edges.

3.1. The θ-Graph is Chromatically Unique.

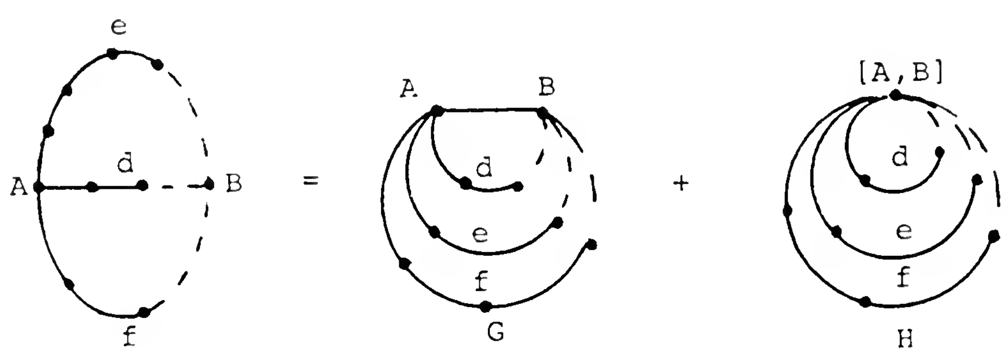
Two graphs G_1 and G_2 are said to be isomorphic if there exists a one-to-one correspondence between their vertices such that two vertices of G_1 are adjacent if and only if the corresponding vertices of G_2 are adjacent. A graph G is chromatically unique if, for all graphs H satisfying $P(H;\lambda) = P(G;\lambda)$, H is isomorphic to G.

Recall definition 8, Section 1.2. A θ-graph, denoted by $\theta_{d,e,f}$, is a connected graph consisting of 3 edge-disjoint paths of lengths d, e, and f, with $d \le e \le f$, joining two vertices of degree 3. All other vertices have degree 2. $\theta_{d,e,f}$ has $n = d + e + f - 1$ vertices and $d + e + f$ edges.

Theorem 3.1.1. $\theta_{d,e,f}$ is chromatically unique.

Two proofs are known. Here we use a clever addition of an edge to make the computation of the chromatic polynomial simpler. It was suggested by the referee who recommended publication of the paper [19] containing this proof. The second is a direct derivation, which appears in [19A].

Proof. To establish the chromatic polynomial of $\theta_{d,e,f}$, we can use Theorem 1.3.1.



For $d > 1$ we add the edge connecting the 2 vertices of degree 3 to obtain 2 graphs, G and H. G consists of 3 cycles with exactly one edge in common, and H consists of 3 cycles with exactly one common vertex. (With $d = 1$, chromatic uniqueness of the θ-graph is shown in Chao and Whitehead [5].)

It is well known that the chromatic polynomial of the cycle of length ℓ is $P(C_\ell;\lambda) = (\lambda - 1)[(\lambda - 1)^{\ell-1} + (-1)^\ell]$.

$$P(\theta_{d,e,f};\lambda) = P(G;\lambda) + P(H;\lambda), \text{ where}$$

$$P(G;\lambda) = \frac{P(C_{d+1};\lambda)\cdot P(C_{e+1};\lambda)\cdot P(C_{f+1};\lambda)}{[\lambda(\lambda - 1)]^2}$$

and

$$P(H;\lambda) = \frac{P(C_d;\lambda)\cdot P(C_e;\lambda)\cdot P(C_f;\lambda)}{\lambda^2}$$

by Theorem 1.3.3. It is important to note that $P(\theta_{d,e,f};\lambda)$ is divisible by $(\lambda - 1)$ but not by $(\lambda - 1)^2$.

Now let Y be a graph such that $P(Y;\lambda) = P(\theta_{d,e,f};\lambda)$. To establish the isomorphism of Y and $\theta_{d,e,f}$, the following properties of Y are necessary and sufficient:

(1) Y is a connected graph with $n + 1$ edges.

Since $\theta_{d,e,f}$ is connected and has $n + 1$ edges, the coefficient of λ^{n-1} in $P(\theta_{d,e,f};\lambda)$ is $n + 1$ and λ^1 has non-zero coefficient. By Properties 2.2.2 and 2.2.4, $P(Y;\lambda) = P(\theta_{d,e,f};\lambda)$ means Y has these properties also.

(2) The degree of every vertex of Y is at least 2.

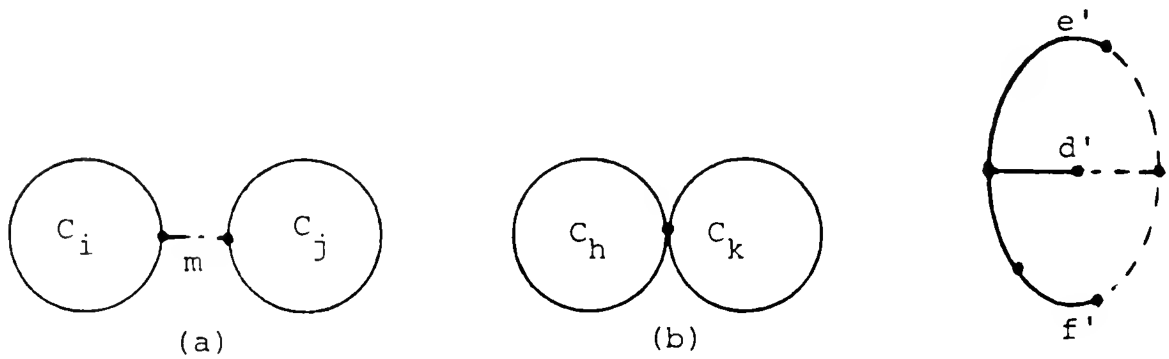
Since Y is connected, no vertex has degree 0. If there exists a vertex v of degree 1, then

$$P(Y;\lambda) = \lambda P(Y';\lambda) - P(Y';\lambda) = (\lambda - 1)P(Y';\lambda) ,$$

where Y' is obtained from Y by deleting the edge joined to v, and identifying v with the vertex to which it was connected. For $n \geq 3$, Y' cannot be colored with just 1 color (it is connected and has at least 2 vertices). Thus $(\lambda - 1)$ is a factor of $P(Y';\lambda)$, making $P(Y;\lambda)$ divisible by $(\lambda - 1)^2$. This is a contradiction.

(3) Y must be a θ-graph.

Since Y has one more edge than vertex and is connected, with no vertices of degree 1, it follows that Y has one of the following three forms:



With form (a), Y consists of 2 cycles, say of lengths i and j, connected by an isthmus of length $m \geq 1$. $n = i + j + m - 1$ and

$$P(Y;\lambda) = \frac{[(\lambda - 1)P(C_i;\lambda)]\cdot[(\lambda - 1)^m P(C_j;\lambda)]}{\lambda(\lambda - 1)} .$$

[Think of 2 cycles with "tails", where a tail of length m adds a factor of $(\lambda - 1)^m$. The tails overlap in K_2.] In this case, $P(Y;\lambda)$ is clearly divisible by $(\lambda - 1)^2$— contradiction.

Form (b) is similar. Here 2 cycles of lengths h and k, $n = h + k - 1$, overlap at one vertex, and

$$P(Y;\lambda) = \frac{P(C_h;\lambda)\cdot P(C_k;\lambda)}{\lambda},$$

again, clearly divisible by $(\lambda - 1)^2$, contradicting what was previously noted.

This leaves only form (c).

(4) If $Y = \theta_{d',e',f'}$ then $d = d'$, $e = e'$, $f = f'$.

So far, Y is connected, has n vertices (2 of degree 3, n - 2 of degree 2), n + 1 edges, and does not contain an isthmus. Therefore Y must be $\theta_{d',e',f'}$ with $d' \le e' \le f'$. Recall $n = d + e + f - 1 = d' + e' + f' - 1$.

Both $\theta_{d,e,f}$ and $Y = \theta_{d',e',f'}$ contain three cycles. The lengths of these cycles can be ordered as follows:

$$d + e \le d + f \le e + f ,$$

$$d' + e' \le d' + f' \le e' + f' .$$

This ordering is a direct consequence of assuming that $d \le e \le f$ and $d' \le e' \le f'$.

<u>Case 1</u>: $d + e \neq d' + e'$.

Let $m = \min\{d + e, d' + e'\}$. Then the coefficient of λ^{n-m+1} in the chromatic polynomials of graphs $\theta_{d,e,f}$ and $\theta_{d',e',f'}$ will be different by Whitney's broken cycle theorem of Section 2.2. One graph will contain a broken cycle of length $m - 1$ while the other graph does not. This contradicts the fact that $P(\theta_{d,e,f};\lambda) = P(\theta_{d',e',f'};\lambda)$.

<u>Case 2</u>: $d + e = d' + e'$ and $d + f \neq d' + f'$.

Let $m = \min\{d + f, d' + f'\}$. Then the coefficient of λ^{n-m+1} in the chromatic polynomials will be different as in Case 1. This is a contradiction.

<u>Case 3</u>: $d + e = d' + e'$ and $d + f = d' + f'$.

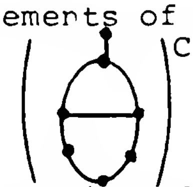
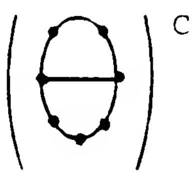
Here, an algebraic argument yields that $d = d'$, $e = e'$ and $f = f'$. Thus $\theta_{d,e,f}$ and $Y = \theta_{d',e',f'}$ are isomorphic, and $\theta_{d,e,f}$ is chromatically unique.

Q.E.D.

3.2. The Graphs $\overline{\theta}_{2,e,f}$ are Chromatically Equivalent.

The investigation of chromatic uniqueness for classes of graphs we know something about led us to consider the complements of such graphs. The θ-graph was no exception. It was simple to program the characteristics of a θ-graph, and generate all of them up to $n = 17$ vertices. The code for this algorithm is found in [19A, Appendix C].Above 17 vertices, additional running time and storage would have been required; results other than those presented here did not appear to be forthcoming; no patterns emerged nor did larger n promise any. Therefore these additions were not implemented.

The output did reveal two interesting facts. First, for any given n, all graphs $\overline{\theta}_{2,e,f}$ are chromatically equivalent, i.e. $P(\overline{\theta}_{2,e,f};\lambda) = P(\overline{\theta}_{2,e',f'};\lambda)$ whenever $e + f = e' + f' \geq 4$. Second, apart from graphs of the form $\overline{\theta}_{2,e,f}$,all θ-graph complements with 17 vertices or less are chromatically distinct, i.e. their chromatic polynomials, amongst all θ-graph complements are different. It may be possible to prove that for all n, θ-graph complements not of the form $\overline{\theta}_{2,e,f}$ are chromatically distinct. No immediate solution to this problem is known.

This is not to say that the complements of θ-graphs are chromatically unique. For example,  and $\overline{\theta}_{1,3,4}$ =  both have the same chromatic polynomial

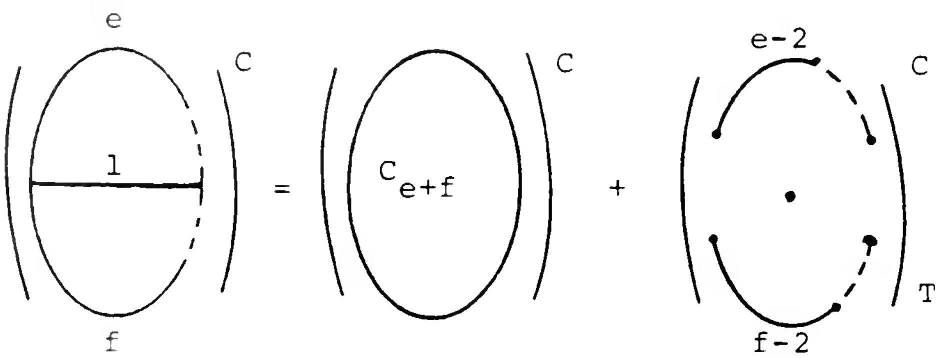
$(\lambda)_7 + 8(\lambda)_6 + 17(\lambda)_5 + 9(\lambda)_4$, yet they are not isomorphic. To see that this is true, we must first establish the following:

Theorem 3.2.1.

$$P(\bar{\theta}_{1,e,f};\lambda) = (\lambda)_n + \sum_{k=1}^{n-1}\left[\frac{n}{k}\binom{k}{n-k} + \sum_{\substack{1\le i\le e\\ 1\le j\le f-1\\ i+j=k}}\left[\binom{i-1}{e-i}\binom{j}{f-1-j}\right]\right](\lambda)_k\ ,$$

$$e,f > 2\ , \qquad e+f = n \geq 6\ .$$

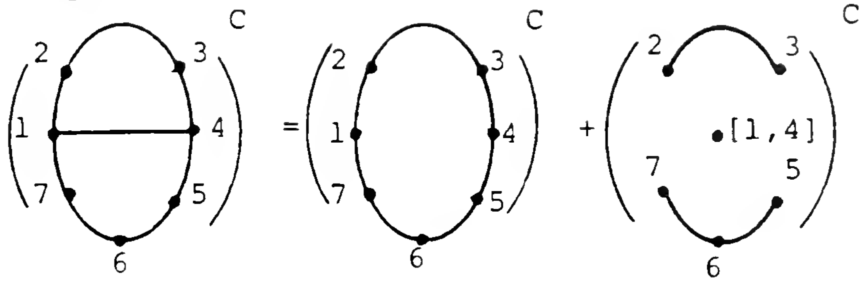
Proof.

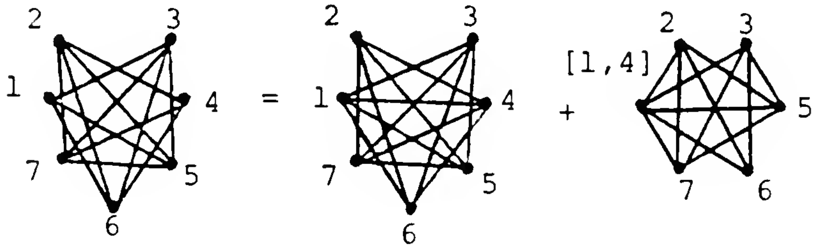


using Theorem 1.3.1

Remark 1. A word is in order here about the application of Theorem 1.3.1 to a graph which is drawn as the complement of a visually simpler graph. When we add an edge to a graph, we delete it in the picture of the complement graph. Also, there will be an edge from a vertex $v_0 \in V$ to the contracted vertex $[v_1,v_2]$ in the contracted complement graph if there was both an edge from v_0 to v_1 and an edge from v_0 to v_2 before the contraction.

For example,





Both equations illustrate the same operation—we are adding edge (1,4) to $\bar{\theta}_{1,3,4}$.

Returning to our proof, the chromatic polynomial of $\bar{\theta}_{1,e,f}$ is equal to the sum of the chromatic polynomial of the complement of a cycle of length e+f and the chromatic polynomial of the Zykov product of a vertex and the complements of two paths of lengths e-2 and f-2, respectively. A path of length r-1 has r vertices. In Chapter 4, we show $P(\bar{P}_r;\lambda) = \sum_{i=0}^{r} \binom{i}{r-i}(\lambda)_i$. In Chapter 5, we show that $P(\bar{C}_r;\lambda) = \sum_{i=1}^{r} \frac{r}{i}\binom{i}{r-i}(\lambda)_i$. Therefore

$$P(\bar{\theta}_{1,e,f};\lambda) = P(\bar{C}_{e+f};\lambda) + P(P_1 \odot \bar{P}_{e-1} \odot \bar{P}_{f-1};\lambda)$$

$$= P(\bar{C}_{e+f};\lambda) + P(P_1;\lambda) \odot P(\bar{P}_{e-1};\lambda) \odot P(\bar{P}_{f-1};\lambda)$$

$$P(\bar{\theta}_{1,e,f};\lambda) = \sum_{k=1}^{n} \frac{n}{k}\binom{k}{n-k}(\lambda)_k$$

$$+ (\lambda)_1 \odot \sum_{i=0}^{e-1}\binom{i}{e-1-i}(\lambda)_i \odot \sum_{j=0}^{f-1}\binom{j}{f-1-j}(\lambda)_j ,$$

$$e,f > 2 .$$

Bring $(\lambda)_1$ inside the second summation:

$$= \sum_{k=1}^{n} \frac{n}{k} \binom{k}{n-k} (\lambda)_k + \sum_{i=0}^{e-1} \binom{i}{e-1-i} (\lambda)_{i+1} \odot \sum_{j=0}^{f-1} \binom{j}{f-j-j} (\lambda)_j \ .$$

Let $i + 1 \to i$, and note that when $j = 0$, we have a term with coefficient equal to zero:

$$= \sum_{k=1}^{n} \frac{n}{k} \binom{k}{n-k} (\lambda)_k + \sum_{i=1}^{e} \binom{i-1}{e-i} (\lambda)_i \odot \sum_{j=1}^{f-1} \binom{j}{f-1-j} (\lambda)_j \ .$$

Combine the Zykov product:

$$= \sum_{k=1}^{n} \frac{n}{k} \binom{k}{n-k} (\lambda)_k + \sum_{\ell=1}^{e+f-1} \sum_{\substack{1 \le i \le e \\ 1 \le j \le f-1 \\ i+j=\ell}} \left[\binom{i-1}{e-i} \binom{j}{f-1-j} \right] (\lambda)_\ell \ .$$

Since $n = e + f$, isolate the term for $k = n$, and combine the remaining coefficients to obtain

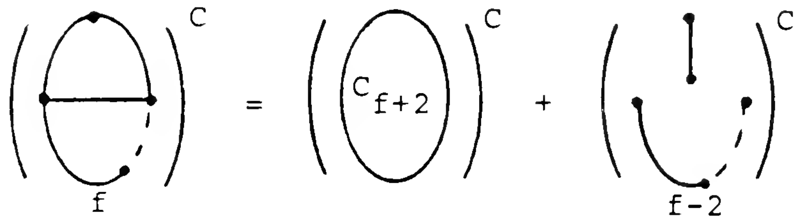
$$P(\overline{\theta}_{1,e,f};\lambda) = (\lambda)_n + \sum_{k=1}^{n-1} \left\{ \frac{n}{k} \binom{k}{n-k} + \sum_{\substack{1 \le i \le e \\ 1 \le j \le f-1 \\ i+j=k}} \left[\binom{i-1}{e-i} \cdot \binom{j}{f-i-j} \right] \right\} (\lambda)_k \ .$$

Q.E.D.

Corollary 3.2.2.

$$P(\overline{\theta}_{1,2,f};\lambda) = (\lambda)_n + (n+1)(\lambda)_{n-1}$$
$$+ \sum_{i=2}^{n-2} \left[\binom{i+1}{n-i} + \binom{i-2}{n-1-i} \right] (\lambda)_i \ , \qquad f > 2 \ ,$$
$$n = f + 2 \ .$$

Proof. This is a degenerate case of $\overline{\theta}_{1,e,f}$. Here we have



$$P(\overline{\theta}_{1,2,f};\lambda) = P(\overline{C}_{f+2};\lambda) + P(\overline{P}_2 \odot \overline{P}_{f-1};\lambda)$$

$$= P(\overline{C}_{f+2};\lambda) + P(\overline{P}_2;\lambda) \odot P(\overline{P}_{f-1};\lambda)$$

$$= \sum_{i=1}^{f+2} \frac{f+2}{i} \binom{i}{f+2-i} (\lambda)_i$$

$$+ [(\lambda)_2 + (\lambda)_1] \odot \sum_{j=0}^{f-1} \binom{j}{f-1-j} (\lambda)_j .$$

Multiply through the Zykov product:

$$= \sum_{i=1}^{f+2} \frac{f+2}{i} \binom{i}{f+2-i} (\lambda)_i + \sum_{j=0}^{f-1} \binom{j}{f-1-j} (\lambda)_{j+2}$$

$$+ \sum_{k=0}^{f-1} \binom{k}{f-1-k} (\lambda)_{k+1} .$$

Since $f > 2$, the $i = 1$ and $k = 0$ terms are zero. Delete them. Let $j + 2 \rightarrow j$, $k + 1 \rightarrow k$, and combine terms:

$$P(\overline{\theta}_{1,2,f};\lambda) = \sum_{i=2}^{f+2} \frac{f+2}{i} \binom{i}{f+2-i} (\lambda)_i + \sum_{j=2}^{f+1} \binom{j-2}{f+1-j} (\lambda)_j$$

$$+ \sum_{k=2}^{f} \binom{k-1}{f-k} (\lambda)_k$$

$$= (\lambda)_{f+2} + [(f+2) + 1](\lambda)_{f+1}$$

$$+ \sum_{i=2}^{f} \left[\frac{f+2}{i} \binom{i}{f+2-i} + \binom{i-2}{f+1-i} + \binom{i-1}{f-i}\right] (\lambda)_i .$$

By Lemma 5.1.2, $\frac{n}{i}\binom{i}{n-i} = \binom{i}{n-i} + \binom{i-1}{n-1-i}$. Using this and Pascal's triangle, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we can simplify the coefficient of $(\lambda)_i$ under the summation:

$$\frac{f+2}{i}\binom{i}{f+2-i} + \binom{i-2}{f+1-i} + \binom{i-1}{f-i}$$

$$= \binom{i}{f+2-i} + \binom{i-1}{f+1-i} + \binom{i-2}{f+1-i} + \binom{i-1}{f-i}$$

$$= \binom{i}{f+2-i} + \binom{i}{f+1-i} + \binom{i-2}{f+1-i}$$

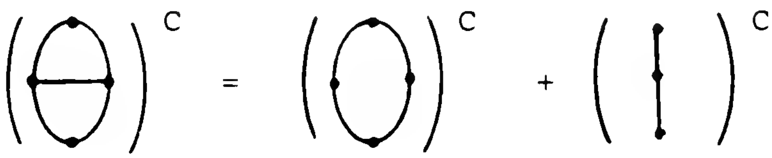
$$= \binom{i+1}{f+2-i} + \binom{i-2}{f+1-i} .$$

Q.E.D.

To complete the analysis:

<u>Corollary 3.2.3</u>. $P(\overline{\theta}_{1,2,2};\lambda) = (\lambda)_4 + 5(\lambda)_3 + 4(\lambda)_2$.

<u>Proof</u>.

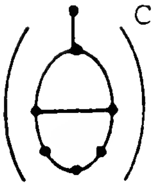


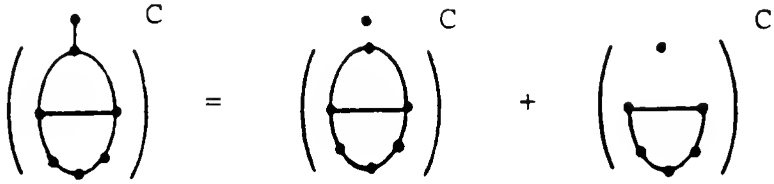
$$= [(\lambda)_4 + 4(\lambda)_3 + 2(\lambda)_2] + [(\lambda)_3 + 2(\lambda)_2]$$

$$= (\lambda)_4 + 5(\lambda)_3 + 4(\lambda)_2 .$$

Q.E.D.

We are now ready to compute the chromatic polynomials of

G:  and $\overline{\theta}_{1,3,4}$.

(1) 

$$P(G;\lambda) = P(P_1 \odot \overline{\theta}_{1,2,4};\lambda) + P(P_1 \odot \overline{C}_5;\lambda)$$

$$= (\lambda)_1 \odot [(\lambda)_6 + 7(\lambda)_5 + 12(\lambda)_4 + 4(\lambda)_3]$$

$$+ (\lambda)_1 \odot [(\lambda)_5 + 5(\lambda)_4 + 5(\lambda)_3]$$

$$= (\lambda)_7 + 8(\lambda)_6 + 17(\lambda)_5 + 9(\lambda)_4 \;;$$

(2) $$P(\overline{\theta}_{1,3,4};\lambda) = (\lambda)_7 + \sum_{k=1}^{6} \left\{\frac{7}{k}\binom{k}{7-k} + \sum_{\substack{1\le i\le 3\\ 1\le j\le 3\\ i+j=k}} \left[\binom{i-1}{3-i}\binom{j}{3-j}\right]\right\}(\lambda)_k$$

$$= (\lambda)_7 + (\lambda)_6\left[\frac{7}{6}\binom{6}{1} + \underset{i=3\ \ j=3}{\binom{2}{0}\binom{3}{0}}\right]$$

$$+ (\lambda)_5\left[\frac{7}{5}\binom{5}{2} + \underset{i=2\ \ j=3}{\binom{1}{1}\binom{3}{0}} + \underset{i=3\ \ j=2}{\binom{2}{0}\binom{2}{1}}\right]$$

$$+ (\lambda)_4\left[\frac{7}{4}\binom{4}{3} + \underset{i=1\ \ j=3}{\binom{0}{2}\binom{3}{0}} + \underset{i=2\ \ j=2}{\binom{1}{1}\binom{2}{1}} + \underset{i=3\ \ j=1}{\binom{2}{0}\binom{1}{2}}\right]$$

$$+ (\lambda)_3\left[\frac{7}{3}\binom{3}{4} + \underset{i=1\ \ j=2}{\binom{0}{2}\binom{2}{1}} + \underset{i=2\ \ j=1}{\binom{1}{1}\binom{1}{2}}\right]$$

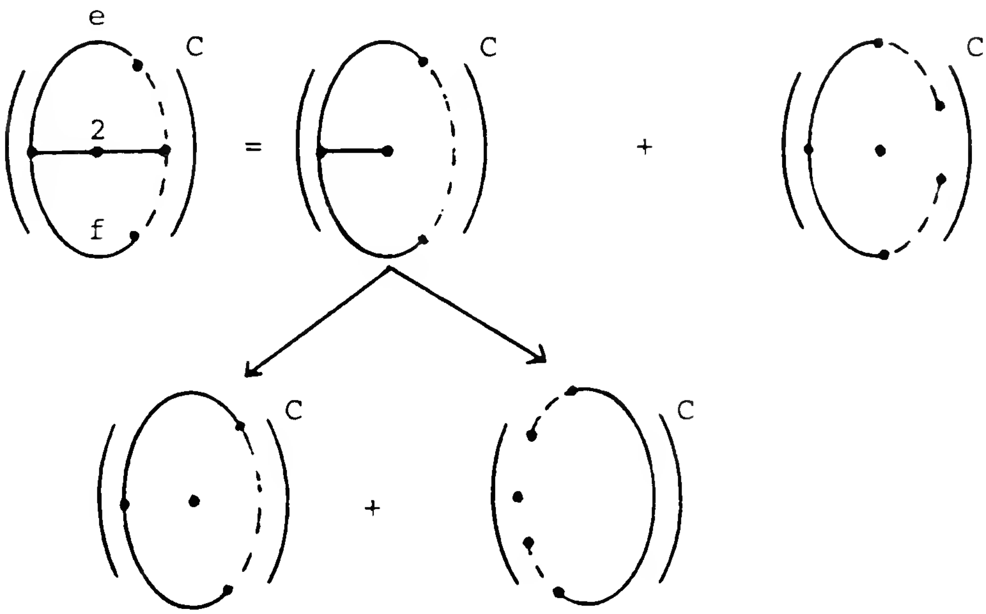
$$+ \ (\lambda)_2 \left[\frac{7}{2} \binom{2}{5} + \underset{i=1}{\binom{0}{2}} \ \underset{j=1}{\binom{1}{2}} \right] + (\lambda)_1 \left[\frac{7}{1} \binom{1}{6} \right]$$

$$= (\lambda)_7 + 8(\lambda)_6 + 17(\lambda)_5 + 9(\lambda)_4 \ .$$

Q.E.D.

Theorem 3.2.4. For any given $n \geq 5$ the graphs $\overline{\theta}_{2,e,f}$ are chromatically equivalent, i.e., $P(\overline{\theta}_{2,e,f};\lambda) = P(\overline{\theta}_{2,e',f'};\lambda)$ whenever $e + f = e' + f'$, $e + f \geq 4$, $n = e + f + 1$.

Proof.



$$P(\overline{\theta}_{2,e,f};\lambda) = P(P_1 \odot \overline{C}_{e+f};\lambda) + 2P(P_1 \odot \overline{P}_{e+f-1};\lambda)$$

$$= (\lambda) \odot [P(\overline{C}_{e+f};\lambda) + 2P(\overline{P}_{e+f-1};\lambda)] \ .$$

From here we can see that the chromatic polynomial of $\overline{\theta}_{2,e,f}$ depends only on the sum $e + f$.

Q.E.D.

Corollary 3.2.5.

$$P(\overline{\theta}_{2,e,f};\lambda) = \sum_{i=1}^{m+1} \left[\binom{i-1}{m+1-i} + \binom{i-2}{m-i} + 2\binom{i-1}{m-i}\right](\lambda)_i ,$$

$$n = m + 1 , \qquad m = e + f \geq 4 .$$

Proof. By Theorem 3.2.4 and Lemma 5.1.2,

$$P(\overline{\theta}_{2,e,f};\lambda) = (\lambda)_1 \odot [P(\overline{C}_{e+f};\lambda) + 2P(\overline{P}_{e+f-1};\lambda)]$$

$$= (\lambda)_1 \odot \left[\sum_{i=1}^{e+f}\left[\binom{i}{e+f-i} + \binom{i-1}{e+f-1-i}\right](\lambda)_i\right.$$

$$\left. + 2\sum_{j=0}^{e+f-1}\binom{j}{e+f-1-j}(\lambda)_j\right]$$

$$= \sum_{i=0}^{e+f}\left[\binom{i}{e+f-i} + \binom{i-1}{e+f-1-i}\right](\lambda)_{i+1}$$

$$+ 2\sum_{j=0}^{e+f-1}\binom{j}{e+f-j-1}(\lambda)_{j+1} .$$

Let $m = e + f$. (Note $m = n - 1$.) Change indices in the first and second sums: $i \leftarrow i + 1$, $j \leftarrow j + 1$, and extend the second sum to $j = e + f + 1$, to get

$$P(\overline{\theta}_{2,e,f};\lambda) = \sum_{i=1}^{m+1} \left[\binom{i-1}{m+1-i} + \binom{i-2}{m-i} + 2\binom{i-1}{m-i}\right](\lambda)_i ,$$

$$m = e + f .$$

Q.E.D.

CHAPTER 4: TREES AND TREE COMPLEMENTS.

4.1. Introduction and Summary of Results.

As defined in Section 1.3, a tree is a connected graph without cycles. Any tree with n vertices, say T_n , has chromatic polynomial $\lambda(\lambda - 1)^{n-1}$; this is proven by Theorem 2.1.2. The converse, Theorem 2.2.1, says that any graph with chromatic polynomial equal to $\lambda(\lambda - 1)^{n-1}$ is a tree. Therefore, any two n-vertex trees are chromatically equivalent. Thus the set of trees fills an equivalence class. Since two nonisomorphic n-vertex trees cannot be distinguished by their chromatic polynomials, we hoped that by examining the chromatic polynomials of their complements, we could establish a method of distinguishing these trees. It might then be possible to make generalizations about chromatic polynomials of graphs and their complements. This idea was based on the heretofore unpublished conjecture by E. Glen Whitehead, Jr.:

Conjecture 4.1.1. Given two nonisomorphic graphs G and H, then either $P(G;\lambda) \neq P(H;\lambda)$ or $P(\bar{G};\lambda) \neq P(\bar{H};\lambda)$ or both.

In Section 4.3, Table 4.3.2, we show this conjecture is false by listing pairs of chromatically equivalent tree complements for nonisomorphic trees with n = 8,9,10 and 11 vertices. Further study may find such pairs for arbitrary $n \geq 11$. Pattern analysis of the examples in Table 4.3.2

did not reveal any generalization. We were able to show the following less dramatic result:

<u>Theorem 4.1.1</u>. All trees with $n \leq 7$ have chromatically distinct complements, i.e., all chromatic polynomials for tree complements with $n \leq 7$ are different.

The first method of proof is by computer generation of all chromatic polynomials of tree complements, $n \leq 11$. For a listing of these chromatic polynomials, see [19A, Appendix B]. The chromatic polynomials for $n \leq 7$ are listed in Table 4.1.1; it is easily seen that all of these tree complements are chromatically distinct.

Since tree complements are dense graphs, CCHROM was used to obtain these polynomials.

All trees with up to 11 vertices were generated via a method published by R. C. Read [23], and discussed in Whitehead [28]. The code for this algorithm can be found in [19A, Appendix D]. Briefly, all trees with n vertices are constructed by adding a new vertex to each tree on n-1 vertices, and connecting this new vertex by an edge to a vertex in T_{n-1} in all possible ways. Each new tree in turn is given a binary 0-1 code such that any two isomorphic trees receive the same code.

This code determines a canonical coding of the tree. Compare this code to those trees that were already generated. If it is a duplicate, an isomorphic tree has already been created; nothing is done. If it is a new code, then the tree is added to the list. The code for each tree is subsequently converted to an edge list, the complement of which then serves as input to CCHROM.

The second proof that all trees with seven vertices or less have chromatically distinct complements comes from the derivation of general formulas for chromatic polynomials of several families of tree complements. A summary of these formulas appears in Table 4.1.2. These formulas were then used for all tree complements $n \leq 7$. Their chromatic polynomials appear in Table 4.1.1. The derivation of these formulas, together with some of their properties, is the content of Section 4.2.

TABLE 4.1.1

Chromatic Polynomials of Tree Complements n = 1,...,7

Note: All are chromatically distinct.

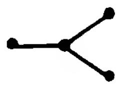
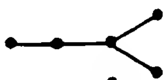
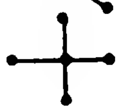
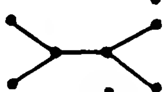
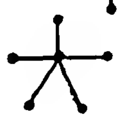
	T_n	$P(\bar{T}_n;\lambda)$	Most Easily Applicable Theorem or Corollary
n = 1		$(\lambda)_1$	4.2.1
n = 2		$(\lambda)_2 + (\lambda)_1$	4.2.1
n = 3		$(\lambda)_3 + 2(\lambda)_2$	4.2.1
n = 4		$(\lambda)_4 + 3(\lambda)_3 + (\lambda)_2$	4.2.1
		$(\lambda)_4 + 3(\lambda)_3$	4.2.3
n = 5		$(\lambda)_5 + 4(\lambda)_4 + 3(\lambda)_3$	4.2.1
		$(\lambda)_5 + 4(\lambda)_4 + 2(\lambda)_3$	4.2.4
		$(\lambda)_5 + 4(\lambda)_4$	4.2.3
n = 6		$(\lambda)_6 + 5(\lambda)_5 + 6(\lambda)_4 + (\lambda)_3$	4.2.1
		$(\lambda)_6 + 5(\lambda)_5 + 5(\lambda)_4 + (\lambda)_3$	4.2.8
		$(\lambda)_6 + 5(\lambda)_5 + 5(\lambda)_4$	4.2.2
		$(\lambda)_6 + 5(\lambda)_5 + 4(\lambda)_4$	4.2.7
		$(\lambda)_6 + 5(\lambda)_5 + 3(\lambda)_4$	4.2.4
		$(\lambda)_6 + 5(\lambda)_5$	4.2.3

TABLE 4.1.1

Continued

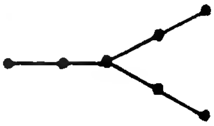
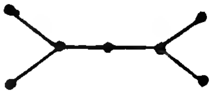
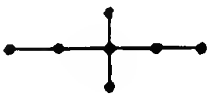
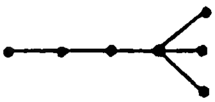
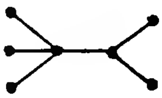
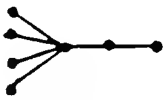
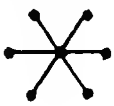
n = 7	T_n	$P(\bar{T}_n;\lambda)$	
		$(\lambda)_7 + 6(\lambda)_6 + 10(\lambda)_5 + 4(\lambda)_4$	4.2.1
		$(\lambda)_7 + 6(\lambda)_6 + 9(\lambda)_5 + 4(\lambda)_4$	4.2.10
		$(\lambda)_7 + 6(\lambda)_6 + 9(\lambda)_5 + 3(\lambda)_4$	4.2.8
		$(\lambda)_7 + 6(\lambda)_6 + 9(\lambda)_5 + 2(\lambda)_4$	4.2.2
		$(\lambda)_7 + 6(\lambda)_6 + 8(\lambda)_5 + 2(\lambda)_4$	4.2.9
		$(\lambda)_7 + 6(\lambda)_6 + 8(\lambda)_5$	4.2.6
		$(\lambda)_7 + 6(\lambda)_6 + 7(\lambda)_5 + 2(\lambda)_4$	4.2.8
		$(\lambda)_7 + 6(\lambda)_6 + 7(\lambda)_5$	4.2.2
		$(\lambda)_7 + 6(\lambda)_6 + 6(\lambda)_5$	4.2.7
		$(\lambda)_7 + 6(\lambda)_6 + 4(\lambda)_5$	4.2.4
		$(\lambda)_7 + 6(\lambda)_6$	4.2.3

TABLE 4.1.2

Chromatic Polynomials of Tree Complements

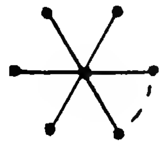
Formula Summary

n = # of vertices, k_i = # of edges

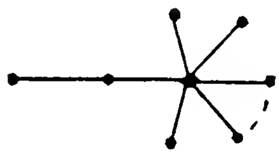
Diagrams depict trees on which complements are based.

$$P(\bar{P}_n;\lambda) = \sum_{i=\lceil n/2\rceil}^{n} \binom{i}{n-i}(\lambda)_i \ , \quad n \geq 0$$

$$P(\bar{K}_{1,n-1};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} \ , \quad n \geq 1$$

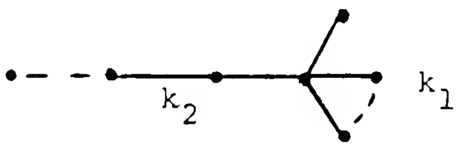


$$P(\bar{R}_n;\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + (n-3)(\lambda)_{n-2} \ , \quad n \geq 3$$

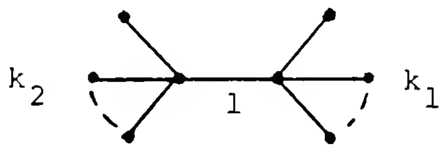


$$P(\bar{F}_{k_1,k_2};\lambda) = \sum_{i=k_1}^{n} \left[\binom{i-k_1}{n-i} + k_1\binom{i-k_1}{n-1-i}\right](\lambda)_i \ , \quad k_1 \geq 0,\ n \geq 1$$

$n = k_1+k_2+1$

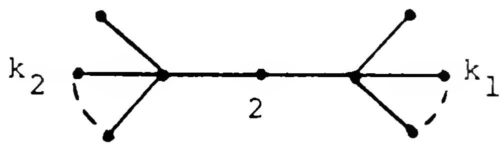


$$P(\bar{F}_{k_1,k_2,1};\lambda) = (\lambda)_n+(n-1)(\lambda)_{n-1}+k_1k_2(\lambda)_{n-2},\quad k_1,k_2 \geq 0$$



$n=k_1+k_2+2$

$$P(\bar{F}_{k_1,k_2,2};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + [(n-3)+k_1k_2](\lambda)_{n-2},\quad k_1,k_2 \geq 0$$

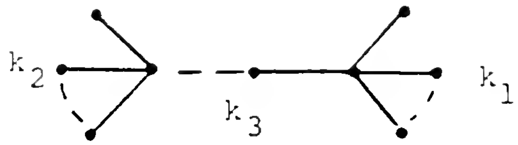


$n=k_1+k_2+3$

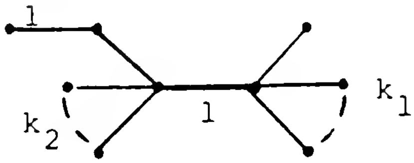
$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=m}^{n} \left[\binom{i-m}{n-i} + m\binom{i-m}{n-1-i} + k_1k_2\binom{i-m}{n-2-i}\right](\lambda)_i \ ,$$

$m=k_1+k_2$; $k_3 \geq 1$, $n = k_1+k_2+k_3+1$.

TABLE 4.1.2
Continued

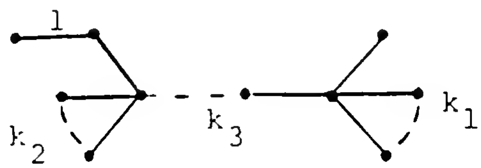


$$P(\bar{F}_{k_1,k_2,1,1};\lambda) = (\lambda)_n+(n-1)(\lambda)_{n-1}+ [k_1+k_2+k_1k_2](\lambda)_{n-2}$$
$$+ k_1(k_2-1)(\lambda)_{n-3} \text{ , } k_1 \geq 0,\ k_2 \geq 1$$



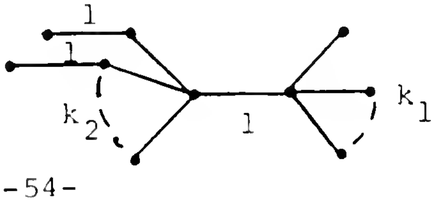
$$P(\bar{F}_{k_1,k_2,k_3,1};\lambda) = \sum_{i=m}^{n} \left[k_1\left\{\binom{i-m}{n-1-i} - \binom{i-m-1}{n-4-i}\right\}\right.$$
$$+ k_2\left\{\binom{i-m-1}{n-1-i} + \binom{i-m}{n-2-i}\right\}$$
$$\left. + k_1k_2\left\{\binom{i-m-1}{n-2-i} + \binom{i-m}{n-3-i}\right\} + \left\{\binom{i-m}{n-i} - \binom{i-m-1}{n-3-i}\right\}\right](\lambda)_i$$

$$k_1 \geq 0,\ k_2 \geq 1,\ k_3 \geq 2,\ m=k_1+k_2,\ n=k_1+k_2+k_3+2$$

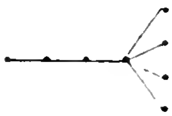
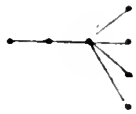
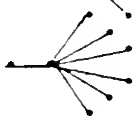


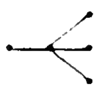
$$P(\bar{F}_{k_1,k_2,1,2};\lambda) = (\lambda)_n+(n-1)(\lambda)_{n-1}+[2k_1+2k_2+k_1k_2+1](\lambda)_{n-2}$$
$$+ [2k_1k_2+k_2-k_1-1](\lambda)_{n-3}+k_1(k_2-2)(\lambda)_{n-4} \text{ ,}$$

$$k_1 \geq 0,\ k_2 \geq 1,\ n=k_1+k_2+4$$



4.2 . Formulas for Chromatic Polynomials of Families of Tree Complements.

Harary [10, page 233] presents a pictorial table of all trees up to 10 vertices. On examining such a list, we notice that trees may be categorized by patterns, or families. For example, on 8 vertices, the trees ,  ,  ,  ,  ,  may be thought of as forks with 1,2,...,6 prongs, respectively, and handles of lengths 6,5,...,1, respectively. Viewing trees as families in this way made it possible to compute chromatic polynomials for many of their complements. Of course there are an infinite number of possible patterns. However, for a fixed, finite integer n, there are a finite number of patterns represented by the trees with n vertices. We chose several so as to place every tree of up to seven vertices into at least one family. This way we have a formula for every such tree complement. Some families do overlap; for example

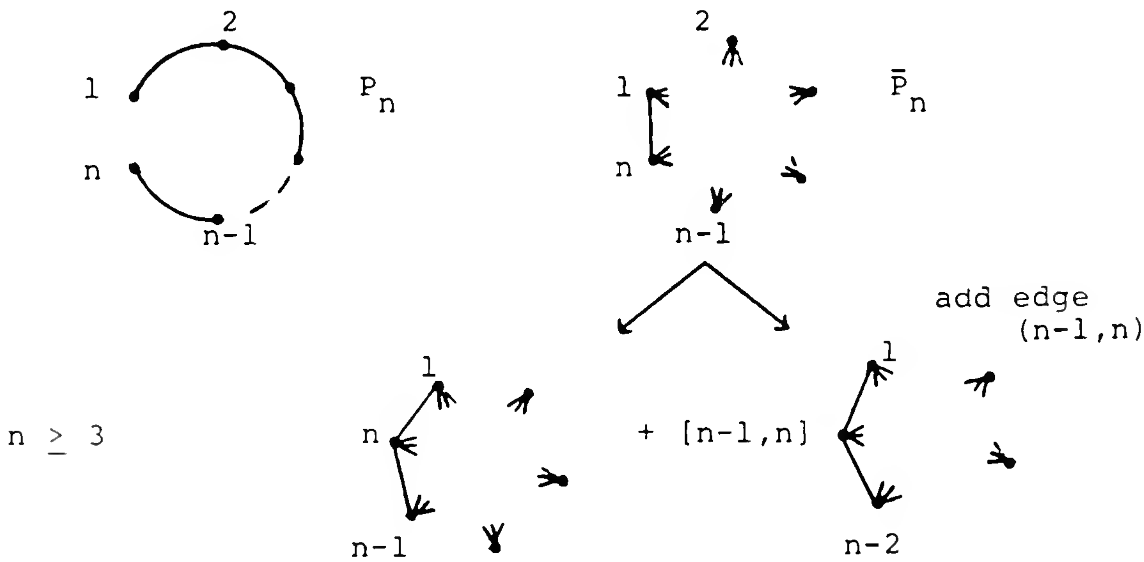


may be considered to be a fork, or a star $K_{1,4}$, which is a complete bipartite graph. These patterns were chosen to fit as many trees as possible; other patterns may also be selected to describe the trees of interest. We categorized the trees themselves rather than their complements because they have less edges and are easier to visualize.

All polynomials will be expressed in the complete graph basis. Since tree complements are dense graphs, their coefficients will be small relative to this basis, in comparison with their coefficients relative to the tree basis, as discussed in Section 2.4. In addition, we may take advantage of Theorems 1.3.3 and 1.3.4.

Theorem 4.2.1. $P(\bar{P}_n;\lambda) = \sum_{i=\lceil n/2\rceil}^{n} \binom{i}{n-i}(\lambda)_i$, $n \geq 0$, where $\lceil x\rceil$ is the smallest integer greater than or equal to x.

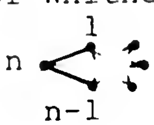
Note: We will first give pictorial motivation of the result, and then verify this by induction. Recall that in these instances the actual drawing of the graph denotes its chromatic polynomial.



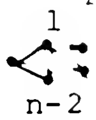
For smaller n these diagrams degenerate. By definition,

$P(\bar{P}_0;\lambda) = 1$, $P(\bar{P}_1;\lambda) = P(P_1;\lambda) = (\lambda)_1$,

$P(\bar{P}_2;\lambda) = (\lambda)_2 + (\lambda)_1$. A vertex labeled [a,b] is the result of identifying the two vertices a and b in one step of Whitney's identity (Theorem 1.3.1). Now



is the Zykov product of a complete graph

on 1 vertex, which is labeled n, and a path complement of length n-1. Similarly [n-1,n] 

is the Zykov product of a complete graph on 1 vertex which is labeled [n,n-1] and a path complement of length n-2. Therefore, by Theorem 1.3.4

$$P(\bar{P}_n;\lambda) = P\big((\lambda)_1 \odot \bar{P}_{n-1};\lambda\big) + P\big((\lambda)_1 \odot \bar{P}_{n-2};\lambda\big)$$

$$= (\lambda)_1 \odot P(\bar{P}_{n-1};\lambda) + (\lambda)_1 \odot P(\bar{P}_{n-2};\lambda)$$

$$= (\lambda)_1 \odot [P(\bar{P}_{n-1};\lambda) + P(\bar{P}_{n-2};\lambda)]$$

The first path complement to fit this recursion occurs at $n = 3$. The coefficients of $P(\bar{P}_n;\lambda)$ can be read from the diagonals of Pascal's triangle, as Table 4.2.1 indicates. Interestingly, the absolute value of the coefficients of the path (or any tree) relative to the null basis form the rows of Pascal's triangle. Table 4.2.1 also illustrates this relationship.

Therefore $P(\bar{P}_n;\lambda) = \displaystyle\sum_{i=\lceil n/2\rceil}^{n} \binom{i}{n-i}(\lambda)_i$, $n \geq 0$.
The lower bound simply reflects the fact that $\binom{i}{n-i} = 0$ if $i < \lceil n/2\rceil$.

<u>Proof</u>: By induction. $\bar{P}_0, \bar{P}_1, \bar{P}_2$, and $\bar{P}_3$ satisfy the polynomial formula. Now suppose the formula is true for all path complements up to k vertices. For k+1 vertices we use the recursion established earlier, namely,

$$P(\bar{P}_{k+1};\lambda) = (\lambda)_1 \odot [P(\bar{P}_k;\lambda) + P(\bar{P}_{k-1};\lambda)] , \qquad k \geq 1$$

$$= (\lambda)_1 \odot \Big[\sum_{i=\lceil k/2\rceil}^{k} \binom{i}{k-i}(\lambda)_i + \sum_{j=\lceil (k-1)/2\rceil}^{k-1} \binom{j}{k-1-j}(\lambda)_j\Big]$$

We can extend both sums down to $i = 0$ and $j = 0$ respectively because all corresponding terms are 0. Consider the second sum.

TABLE 4.2.1

Coefficient Comparison of Chromatic Polynomials of

Paths and Their Complements

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

n	P_n							$\bar{P}_n$						
	λ^7	λ^6	λ^5	λ^4	λ^3	λ^2	λ^1	$(\lambda)_7$	$(\lambda)_6$	$(\lambda)_5$	$(\lambda)_4$	$(\lambda)_3$	$(\lambda)_2$	$(\lambda)_1$
2						1	-1						1	1
3					1	-2	1					1	2	
4				1	-3	3	-1				1	3	1	
5			1	-4	6	-4	1			1	4	3		
6		1	-5	10	-10	5	-1		1	5	6	1		
7	1	-6	15	-20	15	-6	1	1	6	10	4			

We can extend it to include $j = k$, since $\binom{j}{k-1-j} = \binom{k}{-1} = 0$, $k \geq 0$; combining sums,

$$P(\bar{P}_{k+1};\lambda) = (\lambda)_1 \odot \left[\sum_{i=0}^{k} \binom{i}{k-i}(\lambda)_i + \sum_{j=0}^{k} \binom{j}{k-1-j}(\lambda)_j\right]$$

$$= (\lambda)_1 \odot \sum_{i=0}^{k} \left[\binom{i}{k-i} + \binom{i}{k-1-i}\right](\lambda)_i \ .$$

By Pascal's identity, $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$ we obtain

$$P(\bar{P}_{k+1};\lambda) = \sum_{i=0}^{k} \binom{i+1}{k-i}(\lambda)_{i+1} \ ,$$ when the factor $(\lambda)_1$ is brought under the summation.

Now change variables $i+1 \to i$,

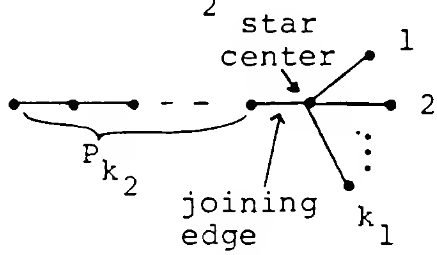
$$P(\bar{P}_{k+1};\lambda) = \sum_{i=1}^{k+1} \binom{i}{k+1-i}(\lambda)_i$$

All terms below $i = \lceil (k+1)/2 \rceil$ will be zero, so that

$$P(\bar{P}_{k+1};\lambda) = \sum_{i=\lceil (k+1)/2 \rceil}^{k+1} \binom{i}{(k+1)-i}(\lambda)_i \ . \qquad \text{Q.E.D.}$$

For $\bar{P}_n$ the coefficient of $(\lambda)_{n-1}$ is $\binom{n-1}{1} = n-1$, which agrees with Eisenberg's formula stated in Section 2.2, namely, $b_{n-1} = \binom{n}{2} - e$. Since $e = \binom{n}{2} - (n-1)$ for all tree complements on n vertices, all their chromatic polynomials will have $(\lambda)_n + (n-1)(\lambda)_{n-1}$ as its first two terms.

We now turn our attention to the family of trees described as forks, and denoted by F_{k_1,k_2}. A <u>fork</u> is a tree consisting of the star K_{1,k_1} attached by an edge to an endpoint of the path P_{k_2} at the star's "center".



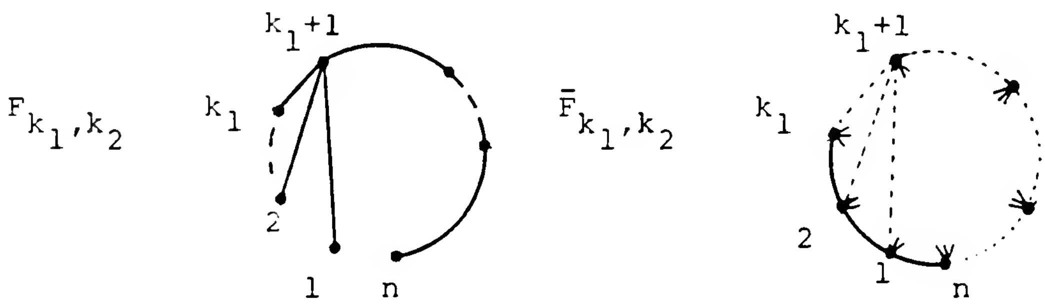
Thus the fork has k_1 prongs and a handle of k_2 edges, with $n = k_1+k_2+1$ vertices.

Theorem 4.2.2.

$$P(\bar{F}_{k_1,k_2};\lambda) = \sum_{i=k_1}^{n} \left[\binom{i-k_1}{n-i} + k_1\binom{i-k_1}{n-1-i}\right](\lambda)_i\,, \quad k_1 \geq 0;\ n \geq 1.$$

Remark. When $k_1 = 0$, this formula reduces to $\sum_{i=0}^{n} \binom{i}{n-i}(\lambda)_i$, precisely the formula for $P(\bar{P}_n;\lambda)$, $n = k_2 + 1$.

Proof. It is easiest to see what $\bar{F}_{k_1,k_2}$ looks like if we rotate F_{k_1,k_2}, and label the vertices as follows:



To obtain the chromatic polynomial of $\bar{F}_{k_1,k_2}$ the following process will be repeated k_1 times:

We break $\bar{F}_{k_1,k_2}$ into two graphs:

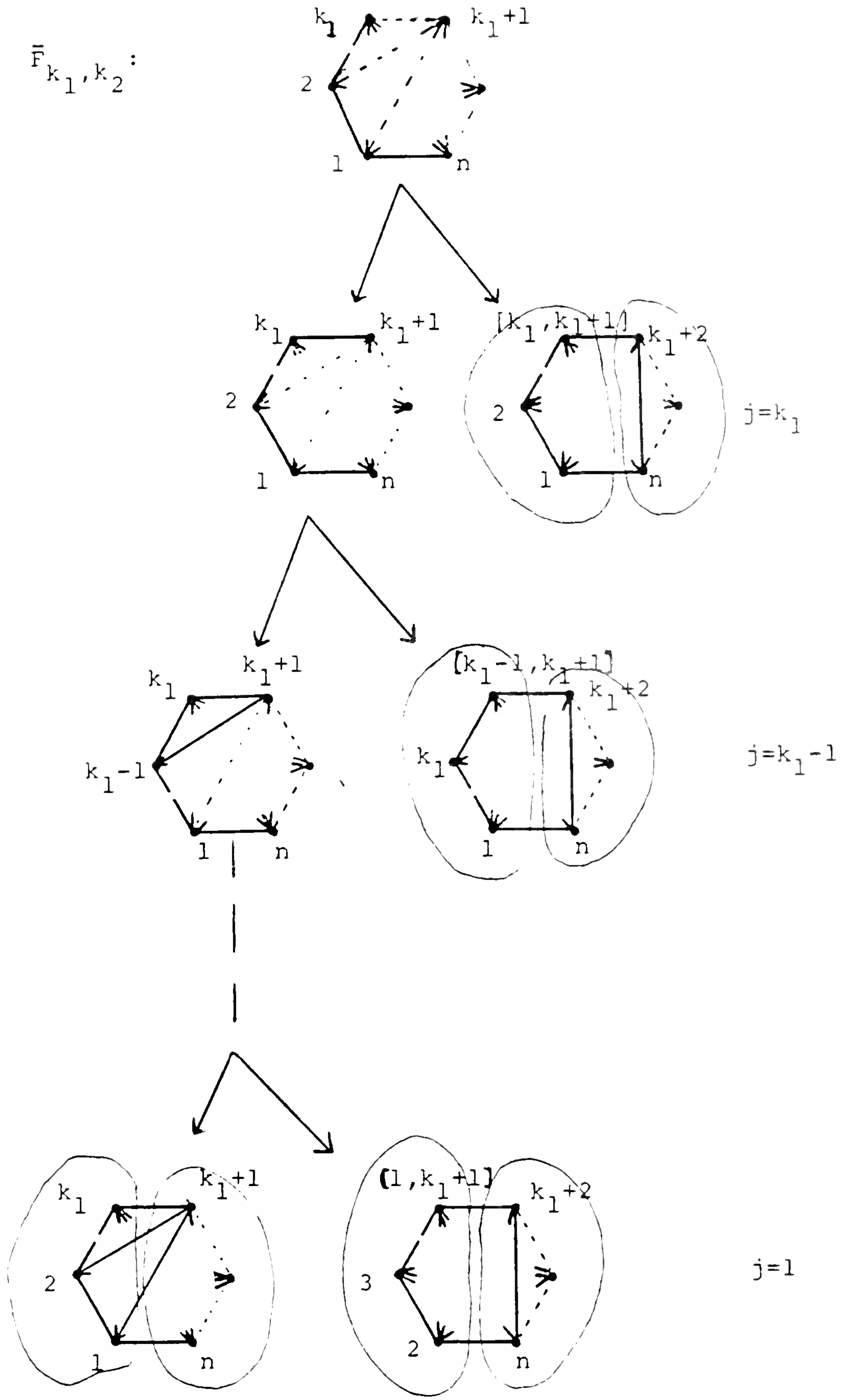
(1) add edge (j, k_1+1), $j = k_1, k_1-1, \ldots, 1$, in turn

(2) identify vertices j and k_1+1

At each stage, graph (1) has n vertices and graph (2) has n-1 vertices. Use the graph with n vertices as input to the next step. By Whitney's identity, the sum of the chromatic polynomials of the final graphs is the chromatic polynomial of the original graph $\bar{F}_{k_1,k_2}$.

Looking at the first two steps and the last step we have:

$\bar{F}_{k_1,k_2}$:



For each of the k_1 steps, graph (2) has the same structure. It is the Zykov product of a complete graph on k_1 vertices and a path complement on $n-(k_1+2)+1 = n-k_1-1 = k_2$ vertices, i.e. $\bar{P}_{k_2} \odot K_{k_1}$. The last step yields a graph of type (1) which can be seen to be the Zykov product of a complete graph on k_1 vertices and a path complement on k_2+1 vertices.

Therefore,

$$P(\bar{F}_{k_1,k_2};\lambda) = P(K_{k_1} \odot \bar{P}_{k_2+1};\lambda) + k_1 P(K_{k_1} \odot \bar{P}_{k_2};\lambda) .$$

By Theorem 1.3.4, this expands to

$$P(\bar{F}_{k_1,k_2};\lambda) = P(K_{k_1};\lambda) \odot P(\bar{P}_{k_2+1};\lambda) + k_1 P(K_{k_1};\lambda) \odot P(\bar{P}_{k_2};\lambda)$$

As shown in Section 1.1, $P(K_{k_1};\lambda) = (\lambda)_{k_1}$, $k_1 \geq 0$. Together this says

$$P(\bar{F}_{k_1,k_2};\lambda) = (\lambda)_{k_1} \odot \sum_{i=\lceil (k_2+1)/2 \rceil}^{k_2+1} \binom{i}{k_2+1-i} (\lambda)_i +$$

$$+ k_1 (\lambda)_{k_1} \odot \sum_{j=\lceil k_2/2 \rceil}^{k_2} \binom{j}{k_2-j} (\lambda)_j .$$

For quick combination of the sums, both can be extended to $i = 0$ and $j = 0$ repsectively, since these lower terms are all zero. When $j = k_2+1$, $\binom{k_2+1}{-1} = 0$, $k_2 \geq 0$, so we can add this term also, to obtain

$$P(\bar{F}_{k_1,k_2};\lambda) = (\lambda)_{k_1} \odot \sum_{i=0}^{k_2+1} \left[\binom{i}{k_2+1-i} + k_1 \binom{i}{k_2-i}\right] (\lambda)_i$$

$$= \sum_{i=0}^{k_2+1} \left[\binom{i}{k_2+1-i} + k_1 \binom{i}{k_2-i}\right] (\lambda)_{i+k_1}$$

Change the running subscript $i+k_1 \to i$, and recall that $n = k_1+k_2+1$:

$$P(\bar{F}_{k_1,k_2};\lambda) = \sum_{i=k_1}^{n} \left[\binom{i-k_1}{n-i} + k_1\binom{i-k_1}{n-1-i}\right](\lambda)_i .$$

Q.E.D.

Remark. We cannot eliminate any terms from this sum, since, if $k_2 = 0$, $(\lambda)_{k_1}$ has nonzero coefficient.

It is easily verified that

(1) If the graph has a single vertex, corresponding to $k_1 = 0$, $k_2 = 0$, $n = 1$, then $P(\bar{F}_{0,0};\lambda) = (\lambda)_1$.

(2) If $k_1 = 1$, the graph becomes $\bar{P}_n$, where $n = k_2+2$, so that

$$P(\bar{F}_{1,k_2};\lambda) = \sum_{i=1}^{n} \left[\binom{i-1}{n-i} + 1\binom{i-1}{n-1-i}\right](\lambda)_i$$

By Pascal's identity,

$$P(\bar{F}_{1,k_2};\lambda) = \sum_{i=1}^{n} \binom{i}{n-i}(\lambda)_i = P(\bar{P}_n;\lambda)$$

Corollary 4.2.3.

$$P(\bar{K}_{1,n-1};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} .$$

Proof. The star complement $\bar{K}_{1,n-1}$ can be looked at as a fork complement with n-2 prongs and handle of length 1. Thus,

$$P(\bar{K}_{1,n-1};\lambda) = P(\bar{F}_{n-2,1};\lambda) = \sum_{i=n-2}^{n}\left[\binom{i-(n-2)}{n-i} + (n-2)\binom{i-(n-2)}{n-1-i}\right](\lambda)_i$$

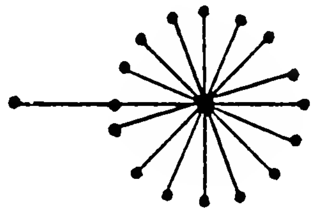
$$= \left[\binom{0}{2} + (n-2)\binom{0}{1}\right](\lambda)_{n-2} + \left[\binom{1}{1} + (n-2)\binom{1}{0}\right](\lambda)_{n-1} + \left[\binom{2}{0} + (n-2)\binom{2}{-1}\right](\lambda)_n$$

$$= (n-1)(\lambda)_{n-1} + (\lambda)_n .$$

Q.E.D.

A sparkler on n vertices, denoted by R_n, is a fork with handle of length 2. It may also be seen as a star on n-1 vertices with an additional edge and vertex joining one of the terminal vertices.

e.g.: R_{18}



Corollary 4.2.4.

$$P(\bar{R}_n;\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + (n-3)(\lambda)_{n-2}$$

Proof. Using $P(\bar{F}_{k_1,k_2};\lambda)$ with $k_2 = 2$, $k_1 = n-3$, we have

$$P(\bar{R}_n;\lambda) = P(\bar{F}_{n-3,2};\lambda) = \sum_{i=n-3}^{n} \left[\binom{i-(n-3)}{n-i} + (n-3)\binom{i-(n-3)}{n-1-i}\right](\lambda)_i$$

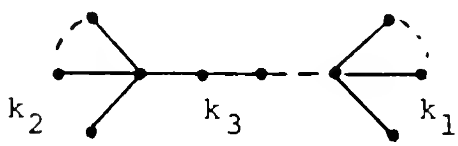
$$= \left[\binom{0}{3} + (n-3)\binom{0}{2}\right](\lambda)_{n-3} + \left[\binom{1}{2} + (n-3)\binom{1}{1}\right](\lambda)_{n-2}$$

$$+ \left[\binom{2}{1} + (n-3)\binom{2}{0}\right](\lambda)_{n-1} + \left[\binom{3}{0} + (n-3)\binom{3}{-1}\right](\lambda)_n$$

Therefore $P(\bar{R}_n;\lambda) = (n-3)(\lambda)_{n-2} + (n-1)(\lambda)_{n-1} + (\lambda)_n$. Q.E.D.

The next family of trees we consider is the one described as a fork with prongs at both ends, and denoted by F_{k_1,k_2,k_3}. A double fork on n vertices has a handle of length k_3 with k_1 prongs at one end and k_2 prongs at the other. Here $n = k_1 + k_2 + k_3 + 1$

F_{k_1,k_2,k_3}



Theorem 4.2.5. Let $m = k_1 + k_2$. Then

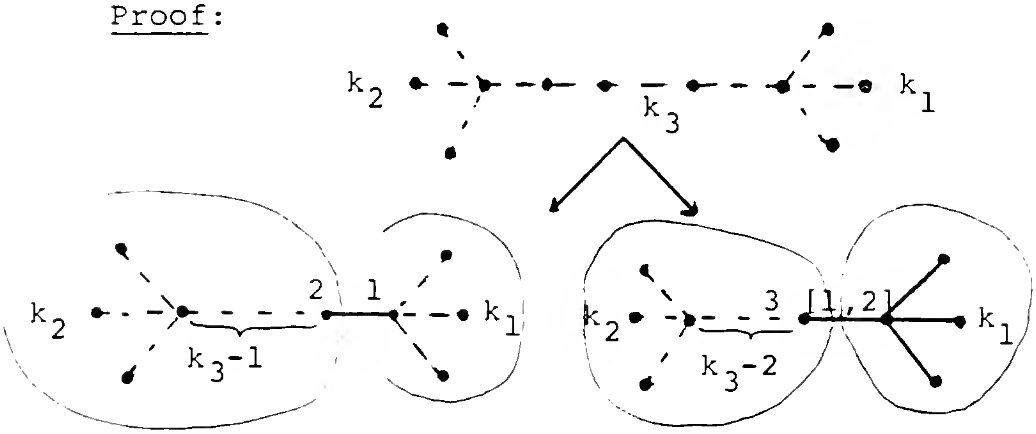
$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=m}^{n}\left[\binom{i-m}{n-i} + m\binom{i-m}{n-1-i} + k_1k_2\binom{i-m}{n-2-i}\right](\lambda)_i \;,$$

$$m \geq 0,\; n \geq 3.$$

Note 1: This formula is symmetric in k_1 and k_2.

Note 2: When $k_2 = 0$, then $m = k_1$, and this formula reduces to chromatic polynomial for the single fork complement $\bar{F}_{k_1,k_3}$.

Proof:



$\bar{F}_{k_1,k_2,k_3}$ where all nondrawn edges exist.

$\bar{F}_{k_2,k_3-1} \odot \bar{K}_{1,k_1}$ $\qquad$ $\bar{F}_{k_2,k_3-2} \odot K_{k_1+1}$ $\qquad$ $k_3 \geq 2$

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = P(\bar{F}_{k_2,k_3-1} \odot \bar{K}_{1,k_1};\lambda) + P(\bar{F}_{k_2,k_3-2} \odot K_{k_1+1};\lambda)$$

$$= P(\bar{F}_{k_2,k_3-1};\lambda) \odot P(\bar{K}_{1,k_1};\lambda) + P(\bar{F}_{k_2,k_3-2};\lambda) \odot P(K_{k_1+1};\lambda)$$

By previously derived formulas:

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \left\{\sum_{i=k_2}^{k_2+k_3}\left[\binom{i-k_2}{k_2+k_3-i} + k_2\binom{i-k_2}{k_2+k_3-1-i}\right](\lambda)_i\right\}$$

$$\odot\left\{(\lambda)_{k_1+1} + k_1(\lambda)_{k_1}\right\}$$

$$+ \left\{\sum_{j=k_2}^{k_2+k_3-1}\left[\binom{j-k_2}{k_2+k_3-1-j} + k_2\binom{j-k_2}{k_2+k_3-2-j}\right](\lambda)_j\right\} \odot (\lambda)_{k_1+1}$$

$$= \sum_{i=k_2}^{k_2+k_3} \left[\binom{i-k_2}{k_2+k_3-i} + k_2\binom{i-k_2}{k_2+k_3-1-i}\right](\lambda)_{i+1+k_1}$$

$$+ \sum_{\ell=k_2}^{k_2+k_3} k_1\left[\binom{\ell-k_2}{k_2+k_3-\ell} + k_2\binom{\ell-k_2}{k_2+k_3-1-\ell}\right](\lambda)_{\ell+k_1}$$

$$+ \sum_{j=k_2}^{k_2+k_3-1} \left[\binom{j-k_2}{k_2+k_3-1-j} + k_2\binom{j-k_2}{k_2+k_3-2-j}\right](\lambda)_{j+1+k_1}$$

we adopt the convention that $\binom{-a}{b} = 0$ for all $a > 0$.

Thus $P(\bar{F}_{k_1,k_2,k_3}; \lambda)$ is broken down into three sums.

(1) Change the running variable in the second sum

$\ell \to \ell+1$ so that it becomes

$$\sum_{\ell=k_2-1}^{k_2+k_3-1} k_1\left[\binom{\ell-k_2+1}{k_2+k_3-\ell-1} + k_2\binom{\ell+1-k_2}{k_2+k_3-\ell-2}\right](\lambda)_{\ell+k_1+1}$$

(2) If the first sum starts at $i = k_2-1$, the added term has coefficient $\left[\binom{-1}{k_3+1} + k_2\binom{-1}{k_3}\right]$, which is always 0, so it does not harm to add it.

(3) Similarly, if the third sum starts at $j = k_2-1$, the added term has coefficient $\left[\binom{-1}{k_3} + k_2\binom{-1}{k_3-1}\right]$, which is 0.

If the sum went to $j = k_2+k_3$ the added term has coefficient $\left[\binom{k_3}{-1} + k_2\binom{k_3}{-2}\right]$, always 0 for $k_3 \geq 2$.

(4) For the second sum, if we add $\ell = k_2+k_3$, its coefficient is $k_1\left[\binom{k_3+1}{-1} + k_2\binom{k_3+1}{-2}\right]$, again 0 for $k_3 \geq 2$.

Thus, if we restrict $k_3 \geq 2$, the chromatic polynomial $P(\bar{F}_{k_1,k_2,k_3}; \lambda)$ simplifies to:

$$\sum_{i=k_2-1}^{k_2+k_3} \left\{\binom{i-k_2}{k_2+k_3-i} + \binom{i-k_2}{k_2+k_3-i-1} + k_1\binom{i-k_2+1}{k_2+k_3-i-1}\right\}$$

$$+ k_2\left\{\binom{i-k_2}{k_2+k_3-i-1} + \binom{i-k_2}{k_2+k_3-i-2} + k_1\binom{i-k_2+1}{k_2+k_3-i-2}\right\}(\lambda)_{i+k_1+1}$$

Using Pascal's identity gives

$$\sum_{i=k_2-1}^{k_2+k_3} \left\{ \binom{i-k_2+1}{k_2+k_3-i} + k_1 \binom{i-k_2+1}{k_2+k_3-i-1} \right\} + k_2 \left\{ \binom{i-k_2+1}{k_2+k_3-i-1} + k_1 \binom{i-k_2+1}{k_2+k_3-i-2} \right\} (\lambda)_{i+k_1+1}$$

Combining terms 2 and 3 yields

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=k_2-1}^{k_2+k_3} \left\{ \binom{i-k_2+1}{k_2+k_3-i} + (k_1+k_2) \binom{i-k_2+1}{k_2+k_3-i-1} + k_1 k_2 \binom{i-k_2+1}{k_2+k_3-i-2} \right\} (\lambda)_{i+k_1+1}$$

Now change the running variable $i \to i-k_1-1$ to get

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=k_1+k_2}^{k_1+k_2+k_3+1} \left\{ \binom{i-(k_1+k_2)}{k_1+k_2+k_3+1-i} + (k_1+k_2) \binom{i-(k_1+k_2)}{k_1+k_2+k_3-i} + k_1 k_2 \binom{i-(k_1+k_2)}{k_1+k_2+k_3-1-i} \right\} (\lambda)_i$$

Recalling $m = k_1+k_2$, $n = k_1+k_2+k_3+1$ yields the final result

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=m}^{n} \left[\binom{i-m}{n-i} + m \binom{i-m}{n-1-i} + k_1 k_2 \binom{i-m}{n-2-i} \right] (\lambda)_i$$

Q.E.D.

<u>Corollary 4.2.6.</u> When $k_3 = 2$, the formula simplifies to

$$P(\bar{F}_{k_1,k_2,2};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + [(n-3)+k_1 k_2](\lambda)_{n-2} .$$

Notice that this is precisely the chromatic polynomial for the sparkler complement $\bar{R}_n$ when $k_2 = 0$.

Proof: $P(\bar{F}_{k_1,k_2,2};\lambda) = \sum_{i=n-3}^{n}\left[\binom{i-(n-3)}{n-i} + (n-3)\binom{i-(n-3)}{n-1-i} + k_1k_2\binom{i-(n-3)}{n-2-i}\right](\lambda)_i$

$$= \left[\binom{0}{3} + (n-3)\binom{0}{2} + k_1k_2\binom{0}{1}\right](\lambda)_{n-3} + \left[\binom{1}{2} + (n-3)\binom{1}{1} + k_1k_2\binom{1}{0}\right](\lambda)_{n-2}$$

$$+ \left[\binom{2}{1} + (n-3)\binom{2}{0} + k_1k_2\binom{2}{-1}\right](\lambda)_{n-1} + \left[\binom{3}{0} + (n-3)\binom{3}{-1} + k_1k_2\binom{3}{-2}\right](\lambda)_n$$

$$= [(n-3) + k_1k_2](\lambda)_{n-2} + (n-1)(\lambda)_{n-1} + (\lambda)_n .$$

Q.E.D.

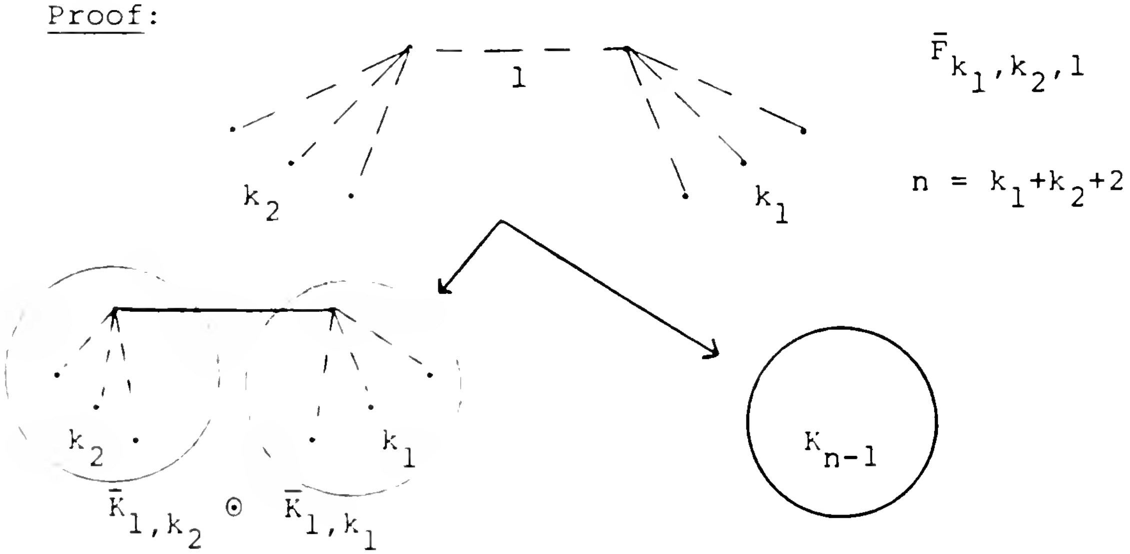
Corollary 4.2.7. When $k_3 = 1$ we have

$$P(\bar{F}_{k_1,k_2,1};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + k_1k_2(\lambda)_{n-2}$$

Remark. The result of this theorem agrees with the general formula for the chromatic polynomial of $\bar{F}_{k_1,k_2,k_3}$. However, we must take this case separately because the proof technique for the general formula requires that $k_3 \geq 2$. This theorem extends the general formula to $k_1,k_2 \geq 0$ and $k_3 \geq 1$.

A direct proof is as follows:

Proof:



Therefore

$$P(\bar{F}_{k_1,k_2,1};\lambda) = P(\bar{K}_{1,k_2} \odot \bar{K}_{1,k_1};\lambda) + P(K_{n-1};\lambda)$$

$$= P(\bar{K}_{1,k_2};\lambda) \odot P(\bar{K}_{1,k_1};\lambda) + P(K_{n-1};\lambda)$$

$$= [(\lambda)_{k_2+1} + k_2(\lambda)_{k_2}] \odot [(\lambda)_{k_1+1} + k_1(\lambda)_{k_1}] + (\lambda)_{k_1+k_2+1}$$

$$= (\lambda)_{k_1+k_2+2} + (k_1+k_2+1)(\lambda)_{k_1+k_2+1} + k_1k_2(\lambda)_{k_1+k_2}$$

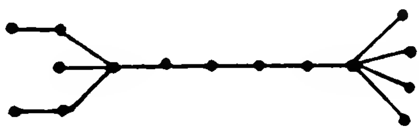
$$= (\lambda)_n + (n-1)(\lambda)_{n-1} + k_1k_2(\lambda)_{n-2} .$$

Q.E.D.

The next family of tree complements for which a chromatic polynomial is computed is an extension of the double fork. Edges are appended to the prongs of the double fork.

An <u>extended double fork</u> F_{k_1,k_2,k_3,k_4}, $1 \le k_4 \le k_2$, $1 \le k_3$, $n = k_1+k_2+k_3+k_4+1$, is a double fork with k_4 additional edges and vertices, where each additional edge and vertex is attached to a different prong among the k_2, e.g.,

$F_{4,3,5,2}$



<u>Theorem 4.2.8.</u>

$$P(\bar{F}_{k_1,k_2,k_3,1};\lambda) = \sum_{i=k_1}^{k_1+k_3+2} \Bigg[k_1 \left\{ \binom{i-k_1}{k_1+k_3+1-i} - \binom{i-1-k_1}{k_1+k_3-2-i} \right\}$$

$$+ k_2 \left\{ \binom{i-k_1-1}{k_1+k_3+1-i} + \binom{i-k_1}{k_1+k_3-i} \right\}$$

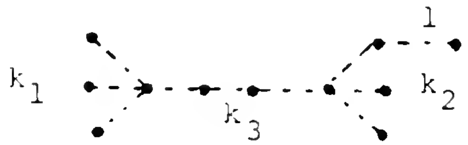
$$+ k_1k_2 \left[\binom{i-k_1-1}{k_1+k_3-i} + \binom{i-k_1}{k_1+k_3-1-i} \right\}$$

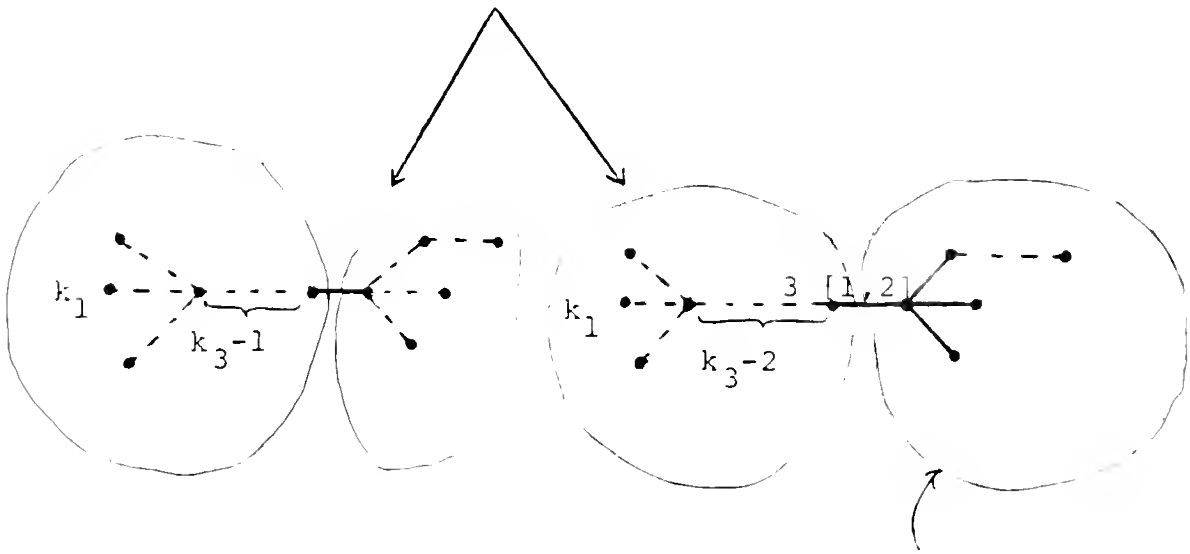
$$+ \left\{ \binom{i-k_1}{k_1+k_3+2-i} - \binom{i-k_1-1}{k_1+k_3-1-i} \right\} \Bigg] (\lambda)_{i+k_2}$$

$$n = k_1+k_2+k_3+2;\ k_1 \ge 0,\ k_2 \ge 1,\ k_3 \ge 2 .$$

Note 1. Since all the extensions occur on the k_2-pronged side of the double fork, this formula is not symmetric in k_1 and k_2.

$\bar{G}=\bar{F}_{k_1,k_2,k_3,1}$:





K_{k_2+2} less 1 edge

$$\bar{F}_{k_1,k_3-1} \odot \bar{R}_{k_2+2} + \bar{F}_{k_1,k_3-2} \odot [(\lambda)_{k_2+2} + (\lambda)_{k_2+1}]$$

$$k_3 \geq 2 \qquad\qquad k_2 \geq 1$$

$$P(\bar{F}_{k_1,k_3-1};\lambda) = \sum_{i=k_1}^{k_1+k_3} \left[\binom{i-k_1}{k_1+k_3-i} + k_1 \binom{i-k_1}{k_1+k_3-1-i}\right](\lambda)_i$$

$$P(\bar{R}_{k_2+2};\lambda) = (\lambda)_{k_2+2} + (k_2+1)(\lambda)_{k_2+1} + (k_2-1)(\lambda)_{k_2}$$

$$P(\bar{F}_{k_1,k_3-2};\lambda) = \sum_{i=k_1}^{k_1+k_3-1} \left[\binom{i-k_1}{k_1+k_3-i-1} + k_1 \binom{i-k_1}{k_1+k_3-2-i}\right](\lambda)_i$$

Therefore,

$$P(\bar{G};\lambda)=\left\{\sum_{i=k_1}^{k_1+k_3}\left[\binom{i-k_1}{k_1+k_3-i}+k_1\binom{i-k_1}{k_1+k_3-1-i}\right](\lambda)_i\right.$$

$$\odot\left[(\lambda)_{k_2+2}+(k_2+1)(\lambda)_{k_2+1}+(k_2-1)(\lambda)_{k_2}\right]$$

$$\left.+\sum_{i=k_1}^{k_1+k_3-1}\left[\binom{i-k_1}{k_1+k_3-1-i}+k_1\binom{i-k_1}{k_1+k_3-2-i}\right](\lambda)_i\right\}\odot[(\lambda)_{k_2+2}+(\lambda)_{k_2+1}]$$

We can extend the second sum to $i = k_1 + k_3$ since that term is zero. Perform the Zykov product and collect like terms to form three sums:

$P(\bar{G};\lambda)$ =

$$(1)\quad \sum_{i=k_1}^{k_1+k_3}\left[\binom{i-k_1}{k_1+k_3-i}+k_1\binom{i-k_1}{k_1+k_3-1-i}+\binom{i-k_1}{k_1+k_3-1-i}\right.$$

$$\left.+k_1\binom{i-k_1}{k_1+k_3-2-i}\right](\lambda)_{i+k_2+2}$$

$$(2)\quad +\sum_{i=k_1}^{k_1+k_3}\left[(k_2+1)\binom{i-k_1}{k_1+k_3-i}+(k_2+1)k_1\binom{i-k_1}{k_1+k_3-1-i}\right.$$

$$\left.+\binom{i-k_1}{k_1+k_3-1-i}+k_1\binom{i-k_1}{k_1+k_3-2-i}\right](\lambda)_{i+k_2+1}$$

$$(3)\quad \sum_{i=k_1}^{k_1+k_3}\left[(k_2-1)\binom{i-k_1}{k_1+k_3-i}+(k_2-1)k_1\binom{i-k_1}{k_1+k_3-1-i}\right](\lambda)_{i+k_2}$$

In sum (1), we use Pascal's identity, extend the sum down to $i = k_1-2$ (adding terms with zero coefficients), and change the running variable $i+2 \rightarrow i$ to get

$$(1)\quad = \sum_{i=k_1}^{k_1+k_3+2}\left[\binom{i-k_1-1}{k_1+k_3+2-i}+k_1\binom{i-k_1-1}{k_1+k_3+1-i}\right](\lambda)_{i+k_2}$$

In sum (2), we add zero terms corresponding to $i = k_1-1$ and $i = k_1+k_3+1$ and change the running variable $i+1 \to i$ to get

$$(2) = \sum_{i=k_1}^{k_1+k_3+2} \left[(k_2+1)\binom{i-k_1-1}{k_1+k_3+1-i} + k_1(k_2+1)\binom{i-k_1-1}{k_1+k_3-i} + \binom{i-k_1-1}{k_1+k_3-i} + k_1\binom{i-k_1-1}{k_1+k_3-1-i}\right](\lambda)_{i+k_2}$$

Similarly, we can extend sum (3) to include terms for $i = k_1 + k_3 + 1$ and $i = k_1 + k_3 + 2$.

Putting this together by grouping like terms gives:

$$\begin{aligned} P(\bar{F}_{k_1,k_2,k_3,1};\lambda) = \sum_{i=k_1}^{k_1+k_3+2} \Bigg\{ & k_1\left[\binom{i-k_1-1}{k_1+k_3+1-i} + \binom{i-k_1-1}{k_1+k_3-i} + \binom{i-k_1-1}{k_1+k_3-1-i} - \binom{i-k_1}{k_1+k_3-1-i}\right] \\ & + k_2\left[\binom{i-k_1-1}{k_1+k_3+1-i} + \binom{i-k_1}{k_1+k_3-i}\right] \\ & + k_1k_2\left[\binom{i-k_1-1}{k_1+k_3-i} + \binom{i-k_1}{k_1+k_3-1-i}\right] \\ & + \left[\binom{i-k_1-1}{k_1+k_3+2-i} + \binom{i-k_1-1}{k_1+k_3+1-i} - \binom{i-k_1}{k_1+k_3-i} + \binom{i-k_1-1}{k_1+k_3-i}\right]\Bigg\}(\lambda)_{i+k_2} \end{aligned}$$

Repeatedly applying Pascal's identity yields the desired result.

Q.E.D.

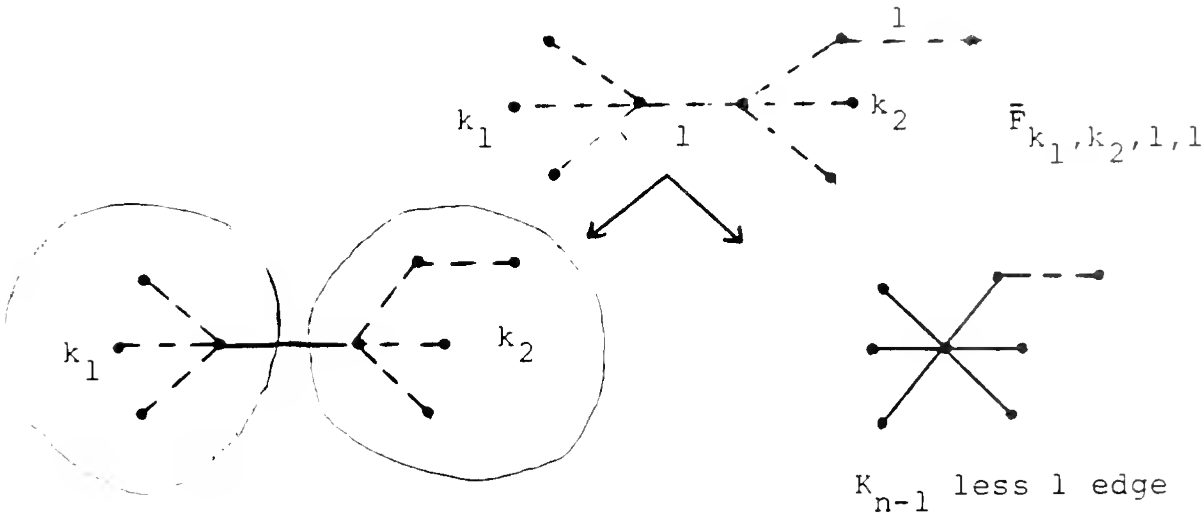
Note 2: When $k_2 = 1$, this formula reduces to that for the single fork with k_1 prongs and handle of length k_3+2.

When the handle of the extended double fork has length 1, Whitney's identities yield a degenerate and simpler result:

Corollary 4.2.9.

$$P(\bar{F}_{k_1,k_2,1,1};\lambda) = (\lambda)_n + (n-1)(\lambda)_{n-1} + [k_1+k_2+k_1k_2](\lambda)_{n-2}$$
$$+ k_1(k_2-1)(\lambda)_{n-3},$$
$$n=k_1+k_2+3,\ k_1 \geq 0,\ k_2 \geq 1.$$

Proof:



$\bar{K}_{1,k_1} \odot \bar{R}_{k_2+2}$ $\qquad$ K_{n-1} less 1 edge

$(\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$

$P(\bar{F}_{k_1,k_2,1,1};\lambda) =$

$$= \left((\lambda)_{k_1+1} + k_1(\lambda)_{k_1}\right) \odot \left((\lambda)_{k_2+2} + (k_2+1)(\lambda)_{k_2+1} + (k_2-1)(\lambda)_{k_2}\right)$$

$$+ (\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$$

$$= (\lambda)_{k_1+k_2+3} + (k_2+1)(\lambda)_{k_1+k_2+2} + (k_2-1)(\lambda)_{k_1+k_2+1} + k_1(\lambda)_{k_1+k_2+2}$$

$$+ k_1(k_2+1)(\lambda)_{k_1+k_2+1} + k_1(k_2-1)(\lambda)_{k_1+k_2} + (\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$$

$$= (\lambda)_{k_1+k_2+3} + [k_2+1+k_1+1](\lambda)_{k_1+k_2+2} + [k_2-1+k_1(k_2+1)+1](\lambda)_{k_1+k_2+1}$$

$$+ k_1(k_2-1)(\lambda)_{k_1+k_2}$$

$$= (\lambda)_n + (n-1)(\lambda)_{n-1} + [k_1+k_2+k_1k_2](\lambda)_{n-2} + k_1(k_2-1)(\lambda)_{n-3}$$

Q.E.D.

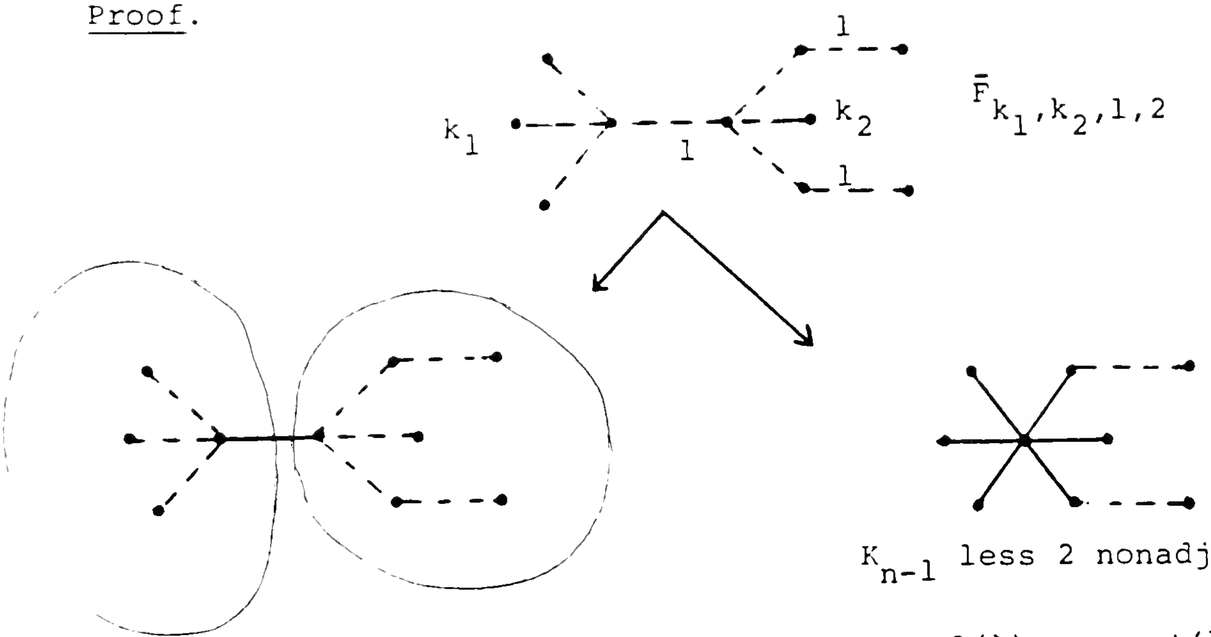
<u>Corollary 4.2.10</u>. $P(\bar{F}_{k_1,k_2,1,2};\lambda) =$

$$= (\lambda)_n + (n-1)(\lambda)_{n-1} + [2k_1+2k_2+k_1k_2+1](\lambda)_{n-2}$$

$$+ [2k_1k_2 + k_2-k_1-1](\lambda)_{n-3} + k_1(k_2-2)(\lambda)_{n-4}$$

$$n = k_1 + k_2 + 4, \quad k_1 \geq 0, \quad k_2 \geq 2.$$

Proof.



$\bar{F}_{k_1,k_2,1,2}$

$\bar{K}_{1,k_1} \odot \bar{F}_{0,k_2-1,2,1}$

K_{n-1} less 2 nonadjacent edges

$(\lambda)_{k_1+k_2+3} + 2(\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$

$$P(\bar{F}_{k_1,k_2,1,2};\lambda)$$

$$= \left((\lambda)_{k_1+1} + k_1(\lambda)_{k_1}\right) \odot \left[\sum_{i=0}^{4} \left\{(k_2-1)\left[\binom{i-1}{3-i}+\binom{i}{2-i}\right] + \binom{i}{4-i} - \binom{i-1}{1-i}\right\}(\lambda)_{i+k_2-1}\right]$$

$$+ (\lambda)_{k_1+k_2+3} + 2(\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$$

$$= \left((\lambda)_{k_1+1} + k_1(\lambda)_{k_1}\right) \odot \left((k_2-2)(\lambda)_{k_2} + (2k_2-1)(\lambda)_{k_2+1}\right.$$

$$\left. + (k_2+2)(\lambda)_{k_2+2} + (\lambda)_{k_2+3}\right) + (\lambda)_{k_1+k_2+3} + 2(\lambda)_{k_1+k_2+2}$$

$$+ (\lambda)_{k_1+k_2+1}$$

$$= (k_2-2)(\lambda)_{k_1+k_2+1} + (2k_2-1)(\lambda)_{k_1+k_2+2} + (k_2+2)(\lambda)_{k_1+k_2+3} +$$

$$+ (\lambda)_{k_1+k_2+4} + k_1(k_2-2)(\lambda)_{k_1+k_2} + k_1(2k_2-1)(\lambda)_{k_1+k_2+1}$$

$$+ k_1(k_2+2)(\lambda)_{k_1+k_2+2} + k_1(\lambda)_{k_1+k_2+3} + (\lambda)_{k_1+k_2+3}$$

$$+ 2(\lambda)_{k_1+k_2+2} + (\lambda)_{k_1+k_2+1}$$

$$= (\lambda)_{k_1+k_2+4} + [k_2+2+k_1+1](\lambda)_{k_1+k_2+3}$$

$$+ [2k_2-1+k_1(k_2+2)+2](\lambda)_{k_1+k_2+2}$$

$$+ [k_2-2+k_1(2k_2-1)+1](\lambda)_{k_1+k_2+1} + k_1(k_2-2)(\lambda)_{k_1+k_2} .$$

Substituting $n = k_1+k_2+4$ yields the indicated result.

Q.E.D.

Table 4.1.1 indicates which theorem or corollary could be used in the calculation of the chromatic polynomials for all tree complements on seven vertices or less. While it is possible to continue this process of obtaining more general formulas, it is clear that these formulas become less and less appealing due to their complexity. It seemed reasonable to stop when we had enough families to include all tree complements on seven vertices or less. As we shall see in Section 4.3, tree complements for $n > 7$ are no longer chromatically distinct.

4.3. Chromatically Distinct and Chromatically Equivalent Tree Complements.

Although all the tree complements with seven vertices or less are chromatically distinct, they are not chromatically unique. Table 4.3.1 lists the 11 tree complements on 7 vertices along with graphs chromatically equivalent to them, discovered on examining Read's list [24]. These diagrams suggested more general statements about the chromatic equivalence of cycles and paths. The results of this investigation are presented in Chapter 5.

The formulas obtained in Section 4.2 can be analyzed in terms of chromatic uniqueness and distinctness. Chromatic uniqueness is difficult to prove; all graphs on n vertices must be brought into consideration. However, the following two theorems reveal distinctness properties.

Theorem 4.3.1. All nonisomorphic fork complements $\bar{F}_{k_1,k_2}$, $n = k_1+k_2+1$, $n \geq 3$ are chromatically distinct.

Proof. $P(\bar{F}_{k_1,k_2};\lambda) = \sum_{i=k_1}^{n} \left[\binom{i-k_1}{n-i} + k_1\binom{i-k_1}{n-1-i}\right](\lambda)_i$.

The coefficient of $(\lambda)_{n-2}$, b_{n-2}, is obtained when the above expression is evaluated at $i = n-2$. We find

$$\binom{n-2-k_1}{2} + k_1\binom{n-2-k_1}{1} = \frac{(n-2-k_1)(n-3-k_1)}{2} + k_1(n-2-k_1)$$

Thus,

$$b_{n-2} = \frac{(n-2-k_1)(n-3+k_1)}{2} .$$

Suppose two different values of k_1 , say p_1 and p_2 , $0 \leq p_1 < p_2$, yield the same value of b_{n-2}. Then

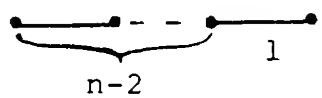
$$\frac{(n-2-p_1)(n-3+p_1)}{2} = \frac{(n-2-p_2)(n-3+p_2)}{2}$$

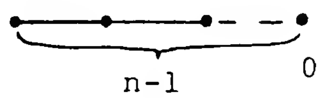
$$\Rightarrow \quad n^2 - 5n + 6 + p_1 - p_1^2 = n^2 - 5n + 6 + p_2 - p_2^2$$

$$p_1 - p_1^2 = p_2 - p_2^2$$

$$p_1(1 - p_1) = p_2(1 - p_2)$$

The only case where these are equal and $p_1 \neq p_2$ is $p_1=0$, $p_2=1$. This corresponds to the complements of





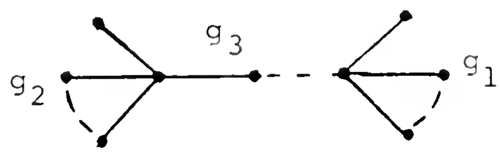
, two isomorphic trees. Therefore all nonisomorphic fork complements, $n \geq 3$, are chromatically distinct.

Q.E.D.

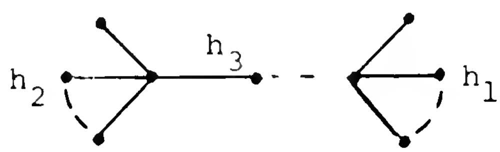
We were also able to establish chromatic distinctness for $\bar{F}_{k_1,k_2,k_3}$, k_3 fixed ≥ 1, where $n = k_1+k_2+k_3+1$, $n \geq 3$.

<u>Theorem 4.3.2</u>. Given two nonisomorphic double fork complements with n vertices and the same length handle, they are chromatically distinct. Let the 2 graphs be $\bar{G}$ and $\bar{H}$.

G:



H:



g_i , $h_i \geq 0$, $i = 1,\ldots,3$; $g_1 \neq h_1$, $g_1 \neq h_2$

$g_2 \neq h_1$, $g_2 \neq h_2$

but $g_3 = h_3$.

Proof . We must show $P(\bar{G};\lambda) \neq P(\bar{H};\lambda)$.

By Theorem 4.2.5,

$$P(\bar{F}_{k_1,k_2,k_3};\lambda) = \sum_{i=m}^{n} \left[\binom{i-m}{n-i} + m\binom{i-m}{n-1-i} + k_1k_2\binom{i-m}{n-2-i}\right](\lambda)_i$$

where $m = k_1+k_2$, $n \geq 3$.

As in Theorem 4.3.1, we examine the coefficient of $(\lambda)_{n-2}$, b_{n-2} , by evaluating the above expression at $i = n-2$ for both $\bar{G}$ and $\bar{H}$.

$$b_{n-2} = \binom{n-2-m}{2} + m\binom{n-2-m}{1} + g_1g_2\binom{n-2-m}{0} = \binom{n-2-m}{2} + m\binom{n-2-m}{1} + h_1h_2\binom{n-2-m}{0}$$

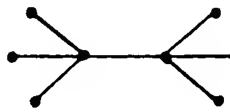
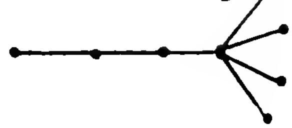
Since $n = g_1+ g_2+ g_3+ 1 = h_1+ h_2+ h_3+ 1$, $g_3 = h_3$, and $m = g_1 + g_2 = h_1+ h_2$, b_{n-2} has the same value in the expressions of $P(\bar{G};\lambda)$ and $P(\bar{H};\lambda)$ unless $g_1g_2 \neq h_1h_2$. Suppose $g_1g_2 = h_1h_2$.

Case 1. $\min\{g_1,g_2\} = 0$. Then $\min\{h_1,h_2\}$ must be zero also. Since $g_1 + g_2 = h_1 + h_2$, this means $\max\{g_1,g_2\} = \max\{h_1,h_2\}$ and the two graphs are isomorphic.

Case 2. $\min\{g_1,g_2\} > 0$. Then $g_1 = h_1h_2/g_2 \Rightarrow h_1h_2/g_2 + g_2 = h_1+h_2 \Rightarrow h_1h_2 + g_2^2 = h_1g_2 + h_2g_2 \Rightarrow h_1h_2 - h_1g_2 = h_2g_2-g_2^2 \Rightarrow h_1(h_2 - g_2) = g_2(h_2- g_2)$. Since $h_2 \neq g_2$, this implies that $h_1 = g_2$. Contradiction. Therefore, $P(\bar{G};\lambda) \neq P(\bar{H};\lambda)$.

Q.E.D.

Table 4.3.2 lists all chromatically equivalent tree complements for n = 8,...,11. This table was generated by the algorithm described in Section 4.1.

Figure 4.3.1 illustrates a direct proof of the chromatic equivalence of G:  C and H:  C

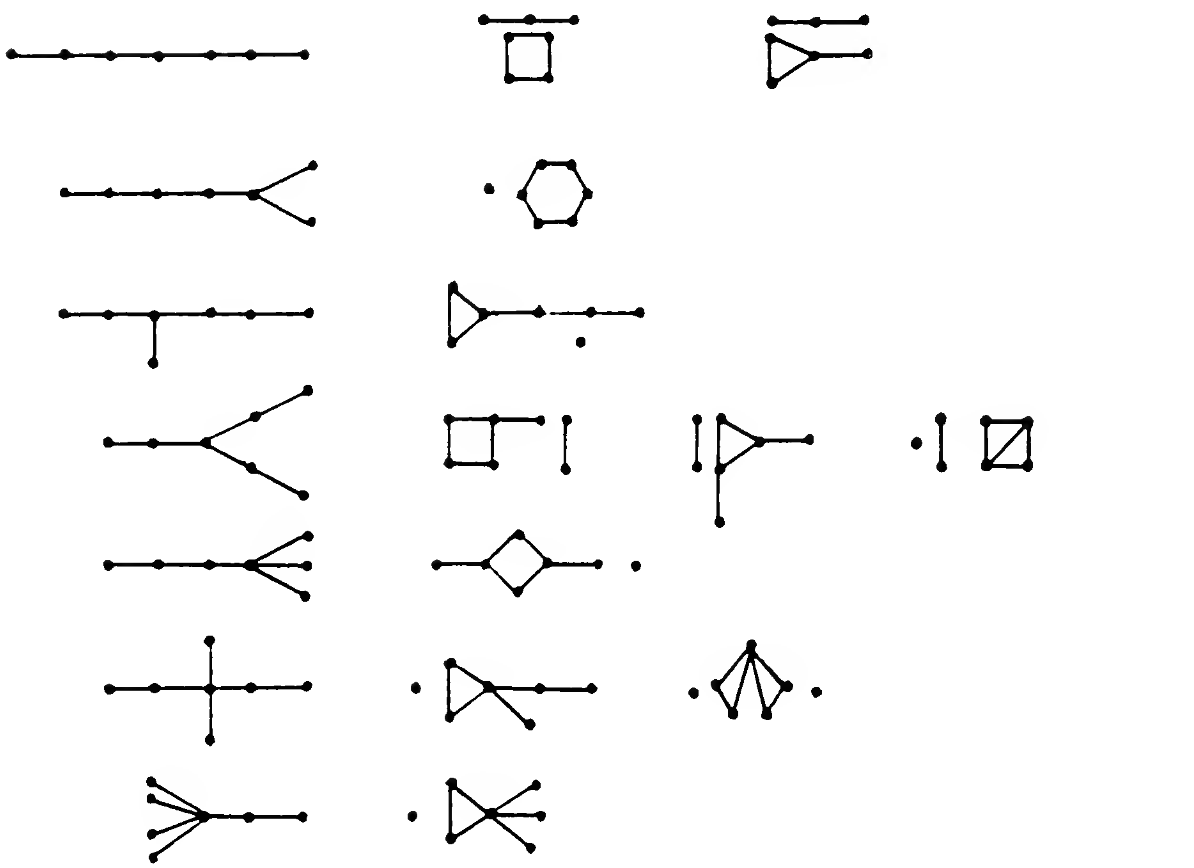
This pair of graphs proves that conjecture 4.1.1 is false. We have found two nonisomorphic graphs whose chromatic polynomials are the same, and whose complements have chromatic polynomials equal to each other. Thus $P(G;\lambda) = P(H;\lambda)$ and $P(\bar{G};\lambda) = P(\bar{H};\lambda)$.

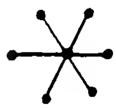
All tree complements for $8 \leq n \leq 11$, other than those listed, are chromatically distinct. At this point, the algorithm terminated due to time limitations.

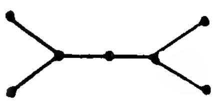
TABLE 4.3.1

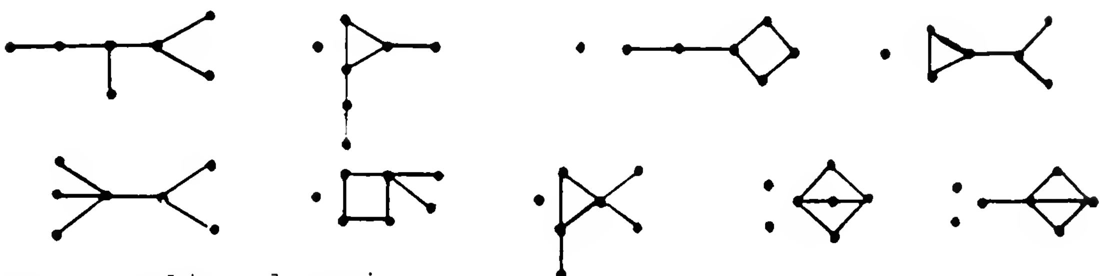
Graphs Chromatically Equivalent to Tree Complements, n = 7

T_7	Chromatically Equivalent Graph All drawn as complements



 CHROMATICALLY UNIQUE

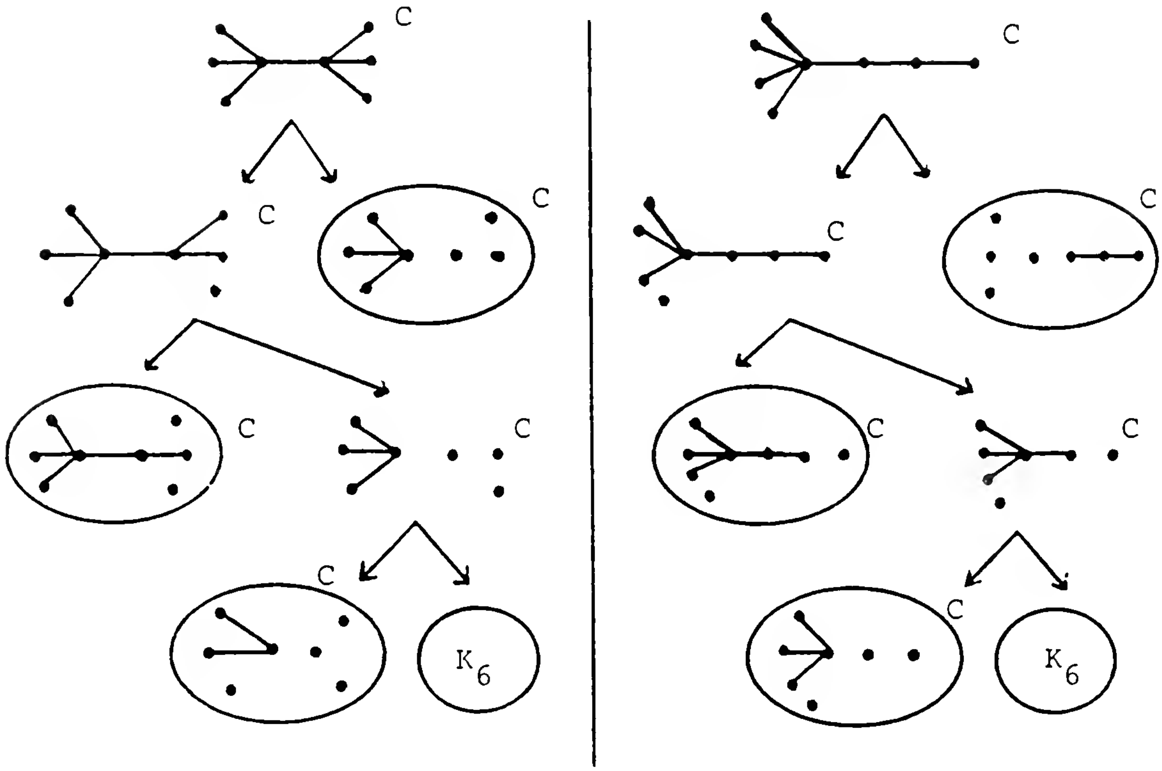
 CHROMATICALLY UNIQUE



Trees are listed as in
Harary [10, p. 233].

FIGURE 4.3.1

Proof of Chromatic Equivalence for the Pair of Tree Complements on 8 Vertices



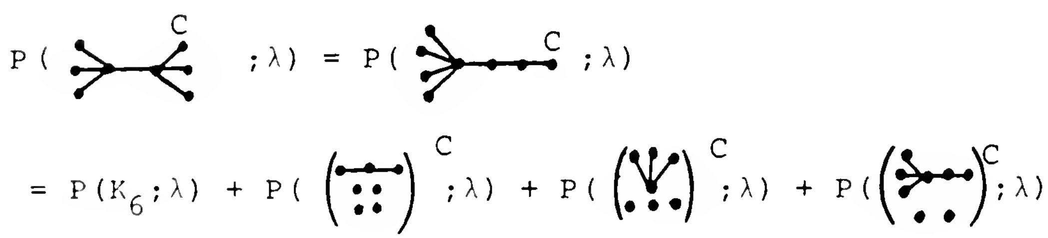


TABLE 4.3.2

Pairs of Tree Complements which are Chromatically Equivalent

$n = 8,9,10,11$

Edge lists are exhibited.

8 vertices — 1 pair

$(\lambda)_8 + 7(\lambda)_7 + 9(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (3,5)\ (2,6)\ (2,7)\ (3,8)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (2,6)\ (2,7)\ (2,8)^C$

9 vertices — 5 pairs

$(\lambda)_9 + 8(\lambda)_8 + 18(\lambda)_7 + 10(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (1,8)\ (5,9)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (4,9)^C$

$(\lambda)_9 + 8(\lambda)_8 + 18(\lambda)_7 + 12(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (2,7)\ (2,8)\ (6,9)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (2,6)\ (3,7)\ (4,8)\ (7,9)^C$

$(\lambda)_9 + 8(\lambda)_8 + 18(\lambda)_7 + 12(\lambda)_6 + 2(\lambda)_5$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (1,8)\ (6,9)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (3,9)^C$

$(\lambda)_9 + 8(\lambda)_8 + 17(\lambda)_7 + 10(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (2,9)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (2,6)\ (2,7)\ (3,8)\ (8,9)^C$

$(\lambda)_9 + 8(\lambda)_8 + 19(\lambda)_7 + 14(\lambda)_6 + 2(\lambda)_5$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (5,9)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (3,8)\ (8,9)^C$

10 vertices — 4 pairs

$(\lambda)_{10} + 9(\lambda)_9 + 21(\lambda)_8 + 12(\lambda)_7$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (1,8)\ (1,9)\ (2,10)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (2,6)\ (2,7)\ (3,8)\ (3,9)\ (4,10)^C$

$(\lambda)_{10} + 9(\lambda)_9 + 21(\lambda)_8 + 9(\lambda)_7$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (1,8)\ (1,9)\ (3,10)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (2,6)\ (2,7)\ (3,8)\ (4,9)\ (4,10)^C$

$(\lambda)_{10} + 9(\lambda)_9 + 26(\lambda)_8 + 28(\lambda)_7 + 9(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (5,9)\ (8,10)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (6,9)\ (9,10)^C$

$(\lambda)_{10} + 9(\lambda)_9 + 26(\lambda)_8 + 27(\lambda)_7 + 8(\lambda)_6$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (2,8)\ (5,9)\ (9,10)^C$

$(1,2)\ (2,3)\ (3,4)\ (4,5)\ (1,6)\ (1,7)\ (3,8)\ (5,9)\ (6,10)^C$

11 vertices — 27 pairs and 2 triples ,

listed in [19A, p. 119-123].

5.1. Cycles and Their Complements.

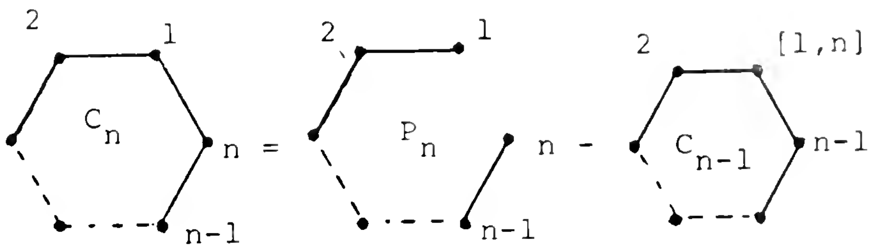
The chromatic polynomial of the cycle is well known.

Theorem 5.1.1 [22]. $P(C_n;\lambda) = (\lambda-1)[(\lambda-1)^{n-1}+(-1)^n]$, $n \geq 3$.

Using Whitney's theorem and recursion, it is easy to show

$$P(C_n;\lambda) = \sum_{i=2}^{n} (-1)^{n-i}\, P(P_i;\lambda)$$

where $\quad P(P_i;\lambda) = \lambda(\lambda-1)^{i-1}$



Notice that the chromatic polynomial of the cycle is derivable exclusively from chromatic polynomials of paths. This relationship will hold in the complements as well.

In calculating the chromatic polynomial of the cycle complement, we will use the following identity for binomial coefficients, which is easily proven.

Lemma 5.1.2.

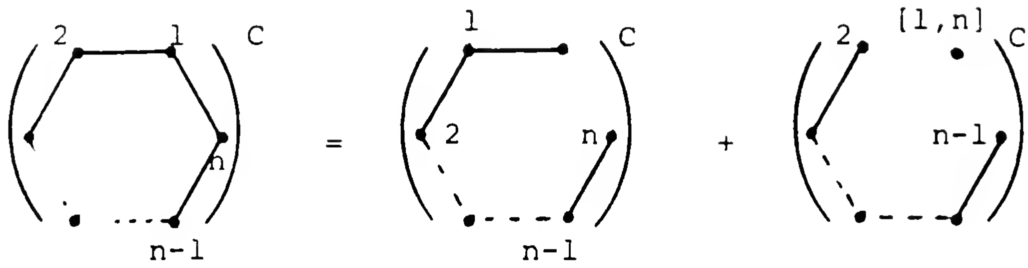
$$\binom{i}{n-i} + \binom{i-1}{n-i-1} = \frac{n}{i}\binom{i}{n-i} \qquad i \geq 1$$

Theorem 5.1.3.

$$P(\bar{C}_n;\lambda) = \sum_{i=\lceil n/2\rceil}^{n} \frac{n}{i}\binom{i}{n-i}(\lambda)_i \, , \quad n \geq 3 \, .$$

Remark 1. Recall that $P(\bar{P}_n;\lambda) = \sum_{i=\lceil n/2\rceil}^{n} \binom{i}{n-i}(\lambda)_i$.

Proof.



$\bar{P}_n$ $\qquad$ $\bar{P}_1 \odot \bar{P}_{n-2}$,

$n \geq 3$

$$P(\bar{C}_n;\lambda) = P(\bar{P}_n;\lambda) + P(\bar{P}_1 \odot \bar{P}_{n-2};\lambda)$$

$$= P(\bar{P}_n;\lambda) + P(\bar{P}_1;\lambda) \odot P(\bar{P}_{n-2};\lambda)$$

$$= \sum_{i=\lceil n/2\rceil}^{n} \binom{i}{n-i}(\lambda)_i + (\lambda)_1 \odot \sum_{j=\lceil n-2\ /2\rceil}^{n-2} \binom{j}{n-2-j}(\lambda)_j$$

To combine these sums easily, extend the sums down to $i,j = 0$. This may be done because all corresponding terms are zero. Now change subscripts $j+1 \to j$. When $j = -1$, and $j = n$, the corresponding terms are also zero. Therefore:

$$P(\bar{C}_n;\lambda) = \sum_{i=0}^{n}\left[\binom{i}{n-i} + \binom{i-1}{n-i-1}\right](\lambda)_i$$

$$= \sum_{i=1}^{n} \frac{n}{i}\binom{i}{n-i}(\lambda)_i \, , \quad \text{by Lemma 5.1.2.}$$

All terms for which $i < \lceil n/2 \rceil$ yield zero coefficients so that

$$P(\bar{C}_n;\lambda) = \sum_{i=\lceil n/2 \rceil}^{n} \frac{n}{i} \binom{i}{n-i} (\lambda)_i \ .$$

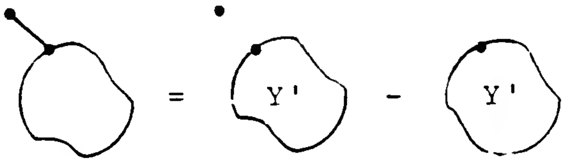
Q.E.D.

This theorem implies that for n a prime, n divides the coefficient $\frac{n}{i}\binom{i}{n-i}$ of $(\lambda)_i$, $\lceil n/2 \rceil \le i \le n$.

Chao and Whitehead [5] prove the following theorem.

<u>Theorem 5.1.4</u>. The cycle C_n , $n \ge 3$, is chromatically unique.

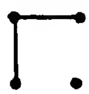
<u>Proof</u>. Suppose there is a graph Y with n vertices such that $P(Y;\lambda) = P(C_n;\lambda)$. Then, by Properties 2.2.2 and 2.2.4, Y is connected and has n edges. No vertex of Y has degree 1, for if it did, $P(Y;\lambda)$ would be divisible by $(\lambda-1)^2$;

Y: 

Y' is connected, with more than 1 vertex — it cannot be colored with 1 color; thus $(\lambda-1)$ is a factor of $P(Y';\lambda)$. Since $P(C_n;\lambda)$ is not divisible by $(\lambda-1)^2$, every vertex in Y has degree at least 2. But Y has only n edges, so that every vertex is of degree exactly 2, and Y is isomorphic to C_n.

Q.E.D.

The chromatic uniqueness of the cycle complement has not been investigated beyond n = 7. $\bar{C}_3 = \bar{K}_3$ is trivially chromatically unique. $\bar{C}_4$ is not chromatically unique since

$\bar{C}_4 =$  $= \lambda^2(\lambda-1)^2 =$  $= P_3 \cup K_1$.

$\bar{C}_5 = C_5$ and is therefore chromatically unique. $\bar{C}_6$ and $\bar{C}_7$ were shown to be chromatically unique by computer-generated listing. There is room for future research in this area.

TABLE ....

GENERATING THE CHROMATIC POLYNOMIAL OF THE CYCLE

FROM THE CHROMATIC POLYNOMIAL OF THE PATH

$$P(C_n;\lambda) = \sum_{i=2}^{n} (-1)^{n-i} P(P_i;\lambda), \qquad n \geq 3.$$

	$(\lambda)_9$	$(\lambda)_8$	$(\lambda)_7$	$(\lambda)_6$	$(\lambda)_5$	$(\lambda)_4$	$(\lambda)_3$	$(\lambda)_2$	$(\lambda)_1$
P_1									1
P_2								1	0
P_3							1	1	0
P_4						1	3	1	0
P_5					1	6	7	1	0
P_6				1	10	25	15	1	0
P_7			1	13	65	90	31	1	0
P_8		1	21	140	350	301	63	1	0
P_9	1	28	246	1050	1701	966	127	1	0
C_3							1	0	
C_4						1	2	1	
C_5					1	5	5	0	
C_6				1	9	20	10	1	
C_7			1	14	56	70	21	0	
C_8		1	20	126	294	231	42	1	
C_9	1	27	246	924	1407	735	85	0	

E.g. $P(C_5;\lambda) = (\lambda)_5 + 6(\lambda)_4 + 7(\lambda)_3 + (\lambda)_2$

$- [(\lambda)_4 + 3(\lambda)_3 + (\lambda)_2]$

$+ (\lambda)_3 + (\lambda)_2$

$- (\lambda)_2$

$= (\lambda)_5 + 5(\lambda)_4 + 5(\lambda)_3$

TABLE ....

GENERATING THE CHROMATIC POLYNOMIAL OF THE CYCLE COMPLEMENT

FROM THE CHROMATIC POLYNOMIAL OF THE PATH COMPLEMENT

$$P(\bar{C}_n;\lambda) = P(\bar{P}_n;\lambda) + {}_1 + P(\bar{P}_{n-2};\lambda), \qquad n \geq 3.$$

	$(\lambda)_9$	$(\lambda)_8$	$(\lambda)_7$	$(\lambda)_6$	$(\lambda)_5$	$(\lambda)_4$	$(\lambda)_3$	$(\lambda)_2$	$(\lambda)_1$
$\bar{P}_1$									1
$\bar{P}_2$								1	1
$\bar{P}_3$							1	2	
$\bar{P}_4$						1	3	1	0
$\bar{P}_5$					1	4	3	0	0
$\bar{P}_6$				1	5	6	1	0	0
$\bar{P}_7$			1	6	10	4	0	0	0
$\bar{P}_8$		1	7	15	10	1	0	0	0
$\bar{P}_9$	1	8	21	20	5	0	0	0	0
$\bar{C}_3$							1	3	0
$\bar{C}_4$						1	4	2	0
$\bar{C}_5$					1	5	5	0	0
$\bar{C}_6$				1	6	9	2	0	0
$\bar{C}_7$			1	7	14	7	0	0	0
$\bar{C}_8$		1	8	20	16	2	0	0	0
$\bar{C}_9$	1	9	27	30	9	0	0	0	0

E.g.: $P(\bar{C}_6;\lambda) = 1(\lambda)_6 + 5(\lambda)_5 + 6(\lambda)_4 + 1(\lambda)_3$

$+ \quad 1(\lambda)_5 + 3(\lambda)_4 + 1(\lambda)_3$

$1(\lambda)_6 + 6(\lambda)_5 + 9(\lambda)_4 + 2(\lambda)_3$

5.2. Cycle Coefficients in the Tree and Complete Graph Bases.

As we saw in Section 5.1,

$$P(C_n;\lambda) = \sum_{i=2}^{n} (-1)^{n-i} P(P_i;\lambda) \quad , \quad n \geq 3 .$$

This, of course, is independent of the basis in which we choose to represent $P(P_i;\lambda)$. In fact, $P(P_i;\lambda) = \lambda(\lambda-1)^{i-1}$ the i^{th} basis element of the tree basis, so that the expression above is the tree basis representation of the cycle.

As we saw in Section 2.1,

$$\lambda(\lambda-1)^{i-1} = \sum_{k=1}^{i} (-1)^{i+k} \binom{i-1}{k-1} \lambda^k$$

$$= \sum_{k=1}^{i} S(i-1,k-1)\,(\lambda)_k$$

Therefore,

in the chromatic polynomial of C_n, the coefficient of λ^k, relative to the null graph basis, is $\sum_{i=2}^{n} (-1)^{i+k} \binom{i-1}{k-1}$, the coefficient of $(\lambda)_k$, relative to the complete graph basis, is $\sum_{i=2}^{n} (-1)^{n-i} S(i-1,k-1)$, while in the tree basis, $\lambda(\lambda-1)^{k-1}$ simply has coefficient $(-1)^{n-k}$.

If n is any prime greater than 3, the coefficients b_i of $P(C_n;\lambda)$ relative to the complete graph basis are divisible by n, $i = 3,\ldots,n-1$. This is true for the coefficients of $P(\bar{C}_n;\lambda)$, as we saw in Section 5.1.

Theorem 5.2.1. If n is a prime, $n > 3$, then n divides b_i, denoted by $n | b_i$, for $3 \leq i \leq n-1$, where

$$P(C_n;\lambda) = \sum_{i=2}^{n} b_i(\lambda)_i , \quad n \geq 3.$$

Proof. For this induction proof we will need Theorem 5.1.1, $P(C_n;\lambda) = (\lambda-1)[(\lambda-1)^{n-1} + (-1)^n]$ and the definition $(\lambda)_k = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-k+1)$. $(\lambda)_k = 0$ whenever $k \geq \lambda+1$. We readily see that $b_1 = 0$.

$$P(C_n;2) = 1[(1)^{n-1}+(-1)^n] = \begin{cases} 0 , & n \text{ odd} \\ 2 , & n \text{ even} \end{cases}$$

$$= b_2(2)_2 + b_1(2)_1 = 2b_2$$

Therefore

$$b_2 = \begin{cases} 0 , & n \text{ odd} \\ 1 , & n \text{ even} \end{cases}$$

For all primes $n > 3$, $b_1 = b_2 = 0$.

$$P(C_n;3) = 2(2^{n-1} + (-1)^n) = b_3(3)_3 + b_2(3)_2 + b_1(3)_1$$

$$2(2^{n-1} + (-1)^n) = 6b_3$$

$$(2^{n-1} + (-1)^n) = 3b_3$$

$$(2^{n-1} - 1) = 3b_3 , \quad \text{for } n \quad \text{odd.}$$

Fermat's Little Theorem [12] states that $a^{p-1} \equiv 1 \pmod p$ for any integer $a \not\equiv 0 \pmod p$. Therefore, $n | (2^{n-1}-1)$ implies $n | 3b_3$, which means $n | b_3$ if $n > 3$.

$$P(C_n;4) = 3(3^{n-1}-1) = b_4(4)_4 + b_3(4)_3 + b_2(4)_2 + b_1(4)_1$$
$$= 24b_4 + 24b_3$$
$$(3^{n-1}-1) = 8b_4 + 8b_3$$

Thus,

$$n \mid (3^{n-1}-1) \text{ and } n \mid b_3 \Rightarrow n \mid 8b_4 \Rightarrow n \mid b_4 \quad \text{if} \quad n > 2,$$

which is certainly true since $n > 3$.

Suppose $n \mid b_3$, $n \mid b_4$, ..., $n \mid b_{k-1}$.

Then

$$P(C_n;k) = (k-1)[(k-1)^{n-1}-1] = b_k(k)_k + b_{k-1}(k)_{k-1} + \cdots$$
$$+ \ldots b_3(k)_3 + b_2(k)_2 + b_1(k)_1.$$

By Fermat's Little Theorem and the induction hypothesis

$$n \mid [(k-1)^{n-1}-1] \quad \text{and} \quad n \mid b_3, \ldots, n \mid b_{k-1} \Rightarrow n \mid \frac{k!}{k-1} b_k .$$

Since k is at most $n-1$, n does not divide $k! \mid (k-1)$.

Q.E.D.

The explicit characterization of b_k in the beginning of this section leads to the next corollary.

<u>Corollary 5.2.2</u>. If n is a prime > 3,

$$n \mid \sum_{i=2}^{n} (-1)^{n-i} S(i-1, k-1) , \quad 3 \leq k \leq n-1.$$

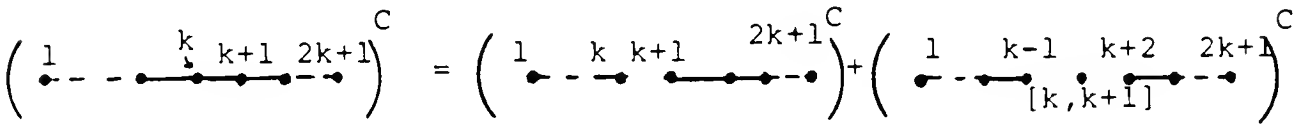
5.3. Combining Path and Cycle Complements.

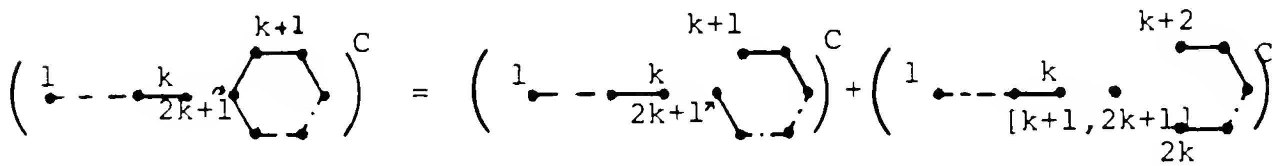
Path and cycle complements can be combined in various ways to yield chromatically equivalent graphs. The following theorems illustrate some of the possibilities.

Theorem 5.3.1.

$$P(\bar{P}_{2k+1};\lambda) = P(\overline{P_k \cup C_{k+1}};\lambda)\ ,\ k \geq 2\ .$$

Proof.





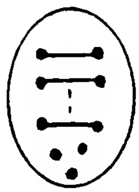
Therefore $P(\bar{P}_{2k+1};\lambda) = P(\overline{P_k \cup C_{k+1}};\lambda)$

$$= P(\bar{P}_k \odot \bar{P}_{k+1};\lambda) + P(\bar{P}_{k-1} \odot P_1 \odot \bar{P}_k;\lambda)$$

If first level decompositions must match, then we are indeed restricted to odd path lengths 2k+1. Suppose not.

Let us compare $\bar{P}_m$ and $\overline{P_n \cup C_{m-n}}$. Using Whitney's identity, add an edge and identify the 2 vertices to produce a first level decomposition. By adding edge $(n,n+1)$ in $\bar{P}_m$, and edge $(m-n-1, m-n)$ in $\overline{P_n \cup C_{m-n}}$, we obtain $\bar{P}_n \odot \bar{P}_{m-n}$ in both cases. By identifying vertices n and $n+1$ in $\bar{P}_m$, we obtain $\bar{P}_{n-1} \odot \bar{P}_1 \odot \bar{P}_{m-n-1}$. In the case of $\overline{P_n \cup C_{m-n}}$, identifying vertices $m-n-1$ and $m-n$ yields $\bar{P}_n \odot \bar{P}_1 \odot \bar{P}_{m-n-2}$. Therefore, since it obviously is not the case that $n-1 = n$, nor $m-n-1 = m-n-2$, the only remaining possibility is $n-1 = m-n-2$ and $n = m-n-1$ i.e. $m = 2n+1$.

In a recent communication E. G. Whitehead Jr. showed the following theorem.

<u>Theorem 5.3.2</u>. Let G:  be the graph with n vertices and k disjoint edges, i.e., G has n-2k isolated vertices. Then $\bar{G}$ is chromatically unique.

<u>Proof</u>. The chromatic number $\chi(\bar{G}) = n-k$. This critical coloring must assign a different color to each of the n-2k vertices that were isolated in G (completely connected in $\bar{G}$), and 1 color for each of the k pairs of vertices that were joined in G (disconnected in $\bar{G}$).

Suppose there exists a graph Y not isomorphic to G, but such that $P(\bar{Y};\lambda) = P(\bar{G};\lambda)$. By Properties 1.3.1 and 2.2.2, this implies that Y has n vertices and k edges.

Case 1. Y has no cycles.

This implies that Y is a forest (a collection of trees) with p components containing edges and n - (k + p) isolated vertices. Since Y and G are nonisomorphic, Y must contain a tree with at least 3 vertices. The vertices of this tree require at least two colors in a coloring of $\bar{Y}$. Thus, in $\bar{Y}$, at least p+1 colors must be used in coloring the p components, and n - (k+p) colors must be used for the vertices isolated in Y. Therefore,

$$\chi(\bar{Y}) \geq (p+1) + (n-(k+p)) = n - k + 1 > \chi(\bar{G}) .$$

Thus, $\bar{Y}$ and $\bar{G}$ are not chromatically equivalent.

Case 2. Y contains at least one cycle.

This implies that Y has p components containing edges, and at least n - (k+p) + 1 isolated vertices. Therefore,

$$\chi(\bar{Y}) \geq p + n - (k + p) + 1 = n - k + 1 > \chi(\bar{G}) .$$

Again, $\bar{Y}$ and $\bar{G}$ are not chromatically equivalent.

Combining Cases 1 and 2, $\bar{G}$ is chromatically unique.

5.4. Wheels and Their Complements.

The wheel W_n can be written as $P_1 \odot C_{n-1}$, $n \geq 4$. Therefore, the following two theorems are easily derived.

Theorem 5.4.1.

$$P(W_n;\lambda) = \sum_{i=1}^{n-2} \sum_{k=1}^{i} (-1)^{n-i-2} S(i,k) (\lambda)_{k+2} , \quad n \geq 4,$$

where $S(i,k)$ is the Stirling number of the second kind.

Proof. $P(W_n;\lambda) = P(P_1 \odot C_{n-1};\lambda) = P(P_1;\lambda) \odot P(C_{n-1};\lambda)$

$$= (\lambda)_1 \odot \sum_{i=1}^{(n-1)-1} \sum_{k=2}^{i+1} (-1)^{(n-1)-i-1} S(i,k-1)(\lambda)_k .$$

Note that in $P(C_{n-1};\lambda)$, k starts with value 2, since a cycle $n \geq 4$, requires at least 2 colors. Multiplying through by $(\lambda)_1$ and letting $k-1 \rightarrow k$,

$$= \sum_{i=1}^{n-2} \sum_{k=1}^{i} (-1)^{n-i-2} S(i,k) (\lambda)_{k+2} .$$

Q.E.D.

Thus, the coefficients of the chromatic polynomial of the wheel are the same as those of the cycle, relative to the complete graph basis, shifted by $(\lambda)_1$.

Theorem 5.4.2.

$$P(W_n;\lambda) = \lambda(\lambda-2)[(\lambda-2)^{n-2} + (-1)^{n-1}] , \quad n \geq 4.$$

Proof. $P(W_n;\lambda) = \lambda P(C_{n-1};\lambda-1)$. This can be easily seen with a coloring argument. First choose the color for the center vertex; we can choose any one of λ colors.

Then proceed to color the cycle C_{n-1} with the $\lambda-1$ remaining colors. This can be done in

$$P(C_{n-1};\lambda-1) = (\lambda-2)[(\lambda-2)^{n-2} + (-1)^{n-1}] \text{ ways.}$$

Therefore

$$P(W_n;\lambda) = \lambda(\lambda-2)[(\lambda-2)^{n-2} + (-1)^{n-1}].$$

Q.E.D.

Theorem 5.4.3.

$$P(W_n;\lambda) = \sum_{j=3}^{n} (-1)^{n-j} \binom{n-1}{j-1} \tau_j + (-1)^{n-2} (n-2)\, \tau_2 \;,\; n \geq 4.$$

where $\tau_j = P(T_j;\lambda) = \lambda(\lambda - 1)^{j-1}$

Proof.

$$P(W_n;\lambda) = \lambda(\lambda-2)^{n-1} + (-1)^{n-1} \lambda(\lambda-2)$$

$$= \lambda(\lambda-1-1)^{n-1} + (-1)^{n-1} \lambda(\lambda-1-1)$$

$$= \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} \lambda(\lambda-1)^{j} + (-1)^{n-1}\lambda(\lambda-1)+(-1)^{n}\lambda$$

$$= \sum_{j=2}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} \lambda(\lambda-1)^{j} + (-1)^{n-2}(n-1)\lambda(\lambda-1)$$

$$+(-1)^{n-1}\lambda + (-1)^{n-1}\lambda(\lambda-1) + (-1)^{n}\lambda$$

$$P(W_n;\lambda) = \sum_{j=2}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} \tau_{j+1} + (-1)^{n-2}(n-2)\tau_2 \;.$$

Changing subscript $j+1 \to j$ yields the desired result.

Q.E.D.

We can easily derive the chromatic polynomial of the wheel complement.

Theorem 5.4.4.

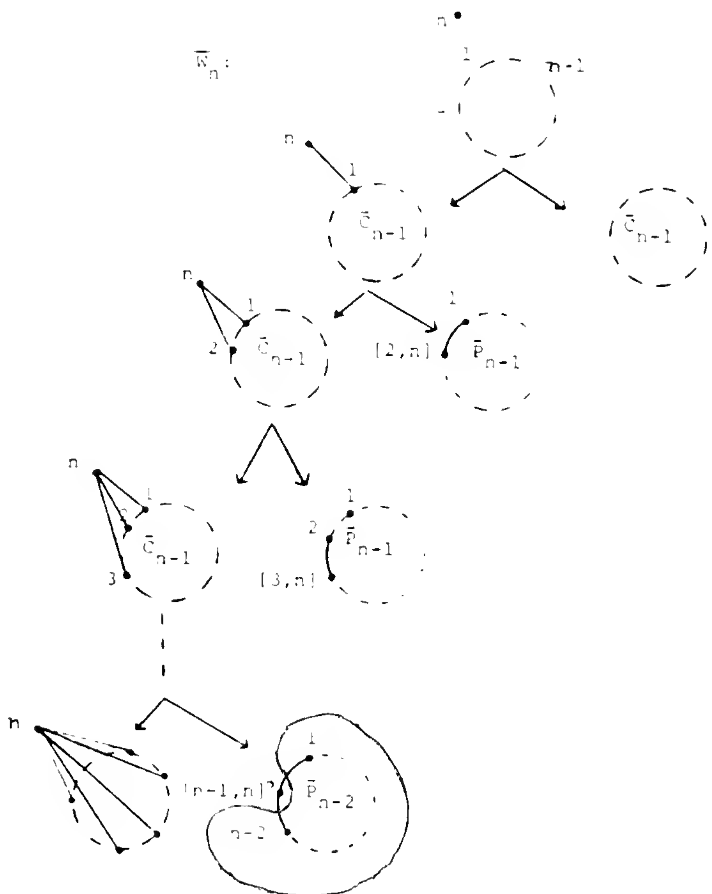
$$P(\bar{W}_n;\lambda) = \sum_{i=\lceil\frac{n-1}{2}\rceil}^{n} \left[\frac{n-1}{i-1}\binom{i-1}{n-i} + \binom{i-1}{n-1-i} + \{n-3+\frac{n-1}{i}\}\binom{i}{n-1-i}\right](\lambda)_i,$$
$$n \geq 4.$$

Proof.

$$P(\bar{W}_n;\lambda) = P(P_1 \cup \bar{C}_{n-1};\lambda) .$$

The proof proceeds via Whitney's identity, by adding the n-1 edges (i,n), i = 1,...,n-1, in turn, and identifying the vertices i and n.

The first step yields a $\bar{C}_{n-1}$. The next n-3 steps each produce a $\bar{P}_{n-1}$. The last step gives a $P_1 \odot \bar{C}_{n-1}$ and a $P_1 \odot \bar{P}_{n-2}$.



$$P(\bar{W}_n;\lambda) = P(P_1 \odot \bar{C}_{n-1};\lambda) + P(P_1 \odot \bar{P}_{n-2};\lambda)$$

$$+ P(\bar{C}_{n-1};\lambda) + (n-3)\, P(\bar{P}_{n-1};\lambda) \qquad n \geq 4$$

Now substitute the results of Sections 4.2 and 5.1, and extend all lower bounds to zero. Change subscripts, and then extend the upper bounds to n if necessary.

$$P(\bar{W}_n;\lambda) = \sum_{i=0}^{n} \left[\frac{n-1}{i-1}\binom{i-1}{n-i} + \binom{i-1}{n-1-i} + \frac{n-1}{i}\binom{i}{n-1-i} + (n-3)\binom{i}{n-1-i}\right](\lambda)_i$$

Combining the third and fourth terms in the expression for the coefficient of $(\lambda)_i$ leads to the desired result.

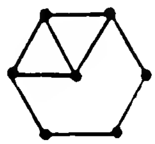
Q.E.D.

5.5. Wheels with Missing Spokes, and Chromatic Uniqueness.

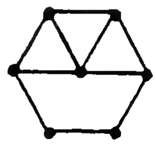
C. Y. Chao and E. G. Whitehead Jr. [6] proved that all members of two infinite families of graphs related to the wheel were chromatically unique. The first family was X_n, $n \geq 5$, where X_n is a wheel W_n with all but 3 consecutive spokes removed.

Z_n, $n \geq 6$, can be defined as the wheel W_n with all but 4 consecutive spokes removed.

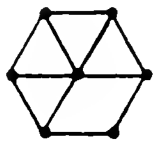
X_7



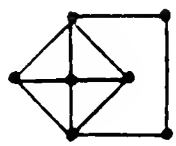
Z_7



Unfortunately, the family of graphs defined as a wheel with all but 5 consecutive spokes removed contains at least one nonchromatically unique graph. Chao and Whitehead showed that



and



are chromatically equivalent.

The same paper showed that W_4 and W_5 were chromatically unique, while W_6 was not. From the work of Section 2.4, computer generated tables demonstrated that W_7 is again chromatically unique. Nothing further is presently known.

In this section we derive a formula for the chromatic polynomial of W_n with all but m consecutive spokes removed, $m = 3,\ldots,n-1$.

Let $W_{n,m}$ denote a wheel on n vertices with all but m consecutive spokes removed. Thus $n-1-m$ spokes are removed.

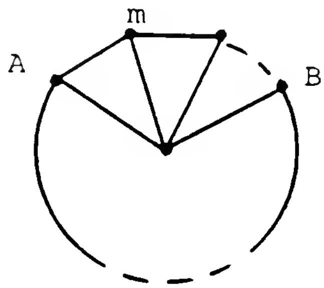
Theorem 5.5.1.

$$P(W_{n,m};\lambda) = (\lambda-2)\Big\{[(\lambda-2)^{m-1}+(-1)^{m}][(\lambda-1)^{n-m}+(-1)^{n-m+1}]$$

$$+ (\lambda-1)[(\lambda-2)^{m-2}+(-1)^{m-1}][(\lambda-1)^{n-m-1}+(-1)^{n-m}]\Big\}$$

$$2 \le m \le n-1, \quad n \ge 5 .$$

Proof. For $3 \le m \le n-2$, edge(A,B) is added, and vertices A and B identified to arrive at the chromatic polynomial.



We then use Theorem 1.3.3 for overlapping graphs.

$$P(W_{n,m};\lambda) = \frac{P(W_{m+1};\lambda)\ P(C_{n-m+1};\lambda)}{\lambda(\lambda-1)} + \frac{P(W_{m};\lambda)\ P(C_{n-m};\lambda)}{\lambda}$$

Using representations for the chromatic polynomial of the wheel and cycle in the null graph basis, we obtain

$$P(W_{n,m};\lambda) = \frac{\lambda(\lambda-2)[(\lambda-2)^{m-1}+ (-1)^{m}](\lambda-1)[(\lambda-1)^{n-m}+ (-1)^{n-m+1}]}{\lambda(\lambda-1)}$$

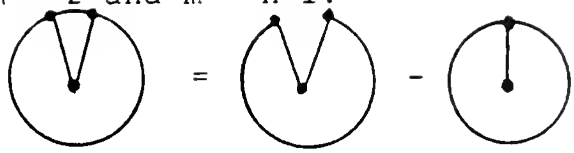
$$+ \frac{\lambda(\lambda-2)[(\lambda-2)^{m-2}+ (-1)^{m-1}](\lambda-1)[(\lambda-1)^{n-m+1}+(-1)^{n-m}]}{\lambda}$$

$$P(W_{n,m};\lambda) = (\lambda-2)\Big[[(\lambda-2)^{m-1}+(-1)^{m}][(\lambda-1)^{n-m}+ (-1)^{n-m+1}]$$

$$+ (\lambda-1)[(\lambda-2)^{m-2}+(-1)^{m-1}][(\lambda-1)^{n-m-1}+(-1)^{n-m}]\Big]$$

Q.E.D.

We can extend the results for $m = 2$ and $m = n-1$.

When $m = 2$ and $n \geq 5$ we have 

$$P(W_{n,2};\lambda) = P(C_n;\lambda) - (\lambda-1)\, P(C_{n-1};\lambda)$$

$$= [(\lambda-1)^n + (-1)^n(\lambda-1)] - [(\lambda-1)^{n-1} + (-1)^{n-2}(\lambda-1)^2]$$

$$= (\lambda-1)^{n-1}(\lambda-2) + (-1)^{n-1}(\lambda-1)[-1 + (\lambda-1)]$$

$$= (\lambda-1)(\lambda-2)[(\lambda-1)^{n-2} + (-1)^{n-1}] .$$

We obtain the same result by letting $m = 2$ in the general formula $P(W_{n,m};\lambda)$.

For completeness, note that

$$P(W_{n,1};\lambda) = (\lambda-1)P(C_{n-1};\lambda) = (\lambda-1)^2[(\lambda-1)^{n-2}+(-1)^{n-1}], \quad n \geq 4.$$

This does not fit into the general formula.

When $m = n-1$, the graph $W_{n,m}$ reduces to the ordinary wheel W_n.

$$P(W_{n,n-1};\lambda) = (\lambda-2)\Big[[(\lambda-2)^{n-2}+(-1)^{n-1}][(\lambda-1)^1+(-1)^2]$$

$$+ (\lambda-1)[(\lambda-2)^{n-3}+(-1)^{n-2}]\,[(\lambda-1)^0+(-1)^1]\Big]$$

$$= \lambda(\lambda-2)[(\lambda-2)^{n-2} + (-1)^{n-1}] .$$

In their unpublished work, Chao and Whitehead used a recursive argument to arrive at the chromatic polynomials of X_n, $n \geq 5$, and Z_n, $n \geq 6$. We obtain closed formulas for $P(X_n;\lambda)$ and $P(Z_n;\lambda)$ directly from $P(W_{n,m};\lambda)$.

$$P(X_n;\lambda) = P(W_{n,3};\lambda) = (\lambda-2)\Big\{[(\lambda-2)^2+(-1)^3][(\lambda-1)^{n-3}+(-1)^{n-2}]$$
$$+(\lambda-1)[(\lambda-2)^1+(-1)^2][(\lambda-1)^{n-4}+ (-1)^{n-3}]\Big\}$$

This simplifies to

$$P(X_n;\lambda) = (\lambda-1)(\lambda-2)[(\lambda-2)(\lambda-1)^{n-3} + 2(-1)^{n-3}] .$$

Also

$$P(Z_n;\lambda) = P(W_{n,4};\lambda) = (\lambda-2)\Big\{[(\lambda-2)^3+(-1)^4][(\lambda-1)^{n-4}+(-1)^{n-3}]$$
$$+ (\lambda-1)[(\lambda-2)^2+ (-1)^3][(\lambda-1)^{n-5}+(-1)^{n-4}] \Big\}$$

This simplifies to

$$P(Z_n;\lambda) = (\lambda-1)(\lambda-2)[(\lambda-1)^{n-4}(\lambda-2)^2+ (-1)^{n-4} (\lambda-4)].$$

Continuing in this vein, we can arrive at formulas for any $W_{n,m}$ with m fixed.

While $W_{n,3}$ and $W_{n,4}$ were shown to be chromatically unique,

$$P(W_{n,5};\lambda) = (\lambda-1)(\lambda-2)[(\lambda-1)^{n-5}(\lambda-2)^3+(-1)^{n-5}(\lambda^2-5\lambda+8)]$$

is not unique for n = 7.

$$P(W_{7,5};\lambda) = \lambda^7 - 11\lambda^6 + 51\lambda^5 - 128\lambda^4 + 184\lambda^3- 143\lambda^2+ 46\lambda.$$

Direct analysis of the general formula did not yield chromatic uniqueness results. This is an open question for future research.

CHAPTER 6: OPEN PROBLEMS

6.1. Open Problems.

Section 6.1 concludes this report with a listing of avenues for further research. Much remains to be done before the characterization of graphs and their chromatic polynomials is complete. Four major problems are as yet unsolved, although some of their subproblems have been successfully analyzed. These four problems are:

1. What are the necessary and sufficient conditions for a polynomial to be a chromatic polynomial?
2. Is the unimodal conjecture true, and if so, relative to which bases is it true?
3. What are the necessary and sufficient conditions for a graph to be chromatically unique?
4. What properties make two graphs chromatically equivalent?

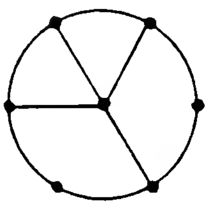
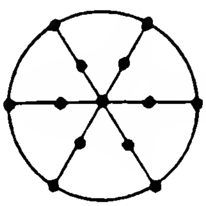
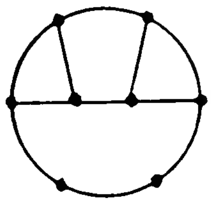
1. What are the necessary and sufficient conditions for a polynomial to be a chromatic polynomial? Certainly the leading coefficient must be equal to one and the polynomial cannot contain a constant term. Relative to the null graph basis, coefficients must alternate in sign. Wilf [32] and Eisenberg [8] have sharpened the upper and lower bounds for the coefficients relative to this basis, but no sufficient conditions have yet been discovered.

2. Is the unimodal conjecture true, and if so, relative to which bases is it true?

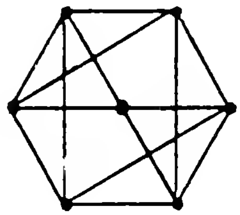
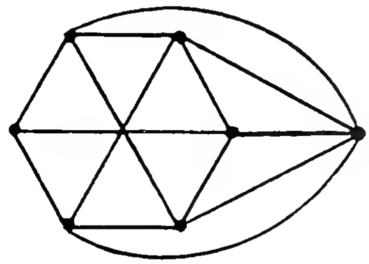
R. C. Read [22] conjectured that the absolute value of the coefficients of the chromatic polynomial of any graph G, relative to the null graph basis, are unimodal. This means the coefficients first increase in absolute magnitude, and then decrease; two successive coefficients may be equal, but there is never one coefficient surrounded by larger coefficients. Read's conjecture allows for any number of equal coefficients.

This conjecture is true for all graphs with seven vertices or less, as verified by computer generation of these polynomials. Of the thousands of chromatic polynomials computed by the author and others, no counterexample has been found. But the truth of the unimodal conjecture, relative to the null graph basis, has not yet been proven or disproven.

3. What are the necessary and sufficient conditions for a graph to be chromatically unique? We know that complete graphs, null graphs, cycles, θ-graphs, $W_{n,3}$ and $W_{n,4}$ are chromatically unique. We know the wheels up to and including W_7 but not W_6 are chromatically unique, as are $\bar{C}_n$, n = 3, 5, 6, and 7; chromatic uniqueness also holds for the complement of the matching graph. These results have yet to be extended. We can also explore other wheel-like graphs—those whose missing spokes are not consecutive, those whose spokes have lengths greater than one, or those with multiple centers, for example.

Graphs such as  ,  , or  are possible extensions of the wheel problem.

Other families of graphs, such as bipartite, complete bipartite, and regular graphs, those graphs for which every vertex has the same degree, have been conjectured to be chromatically unique. On examining the list of chromatic polynomials generated for seven vertices, this last conjecture was found to be false; the graphs

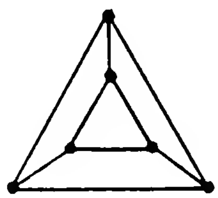
G:  and H: 

both have chromatic polynomial $(\lambda)_7 + 7(\lambda)_6 + 15(\lambda)_5 + 10(\lambda)_4 + 2(\lambda)_3$; G is a regular graph of degree four on seven vertices.

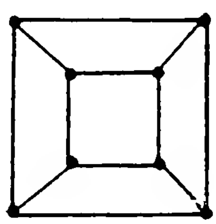
A special family of regular graphs is the prism J_n, n even, $n \geq 6$. This regular graph of degree 3 consists of two cycles, labeled $1,\ldots,\frac{n}{2}$, and $\frac{n}{2} + 1,\ldots,n$, respectively, joined by edges $\{(i,\frac{n}{2} + i), i = 1,\ldots,\frac{n}{2}\}$. Thus, $e = \frac{3}{2}n$.

e.g.:

J_6



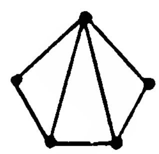
J_8



From as yet unpublished tables of chromatic polynomials for $n \leq 6$ by R. Giudici and R. Vinke, one observes that J_6 is chromatically unique, as are all regular, connected graphs with degree greater than one, for $n \leq 6$. For larger n, chromatic uniqueness of the prism has not been proven or disproven.

Though some sufficiency conditions for chromatic uniqueness exist, necessary conditions other than those mentioned in this manuscript are a much harder problem.

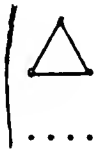
A graph G, with $\chi(G) = k$, is <u>uniquely colorable</u> if every k-coloring partitions the vertex set in the same way. In this case, the coefficient $b_{\chi(G)}$ of $P(G;\lambda)$ relative to the complete graph basis is equal to 1.

e.g.: G:  $\chi(G) = 3$,

$$P(G;\lambda) = (\lambda)_5 + 3(\lambda)_4 + (\lambda)_3$$

G is uniquely colorable.

Is there a relationship between uniquely colorable graphs and chromatically unique graphs? One can easily show

P(()^C ; λ) = P(()^C ; λ) $= (\lambda)_7 + 3(\lambda)_6 + (\lambda)_5$ so that

uniquely colorable graphs are not necessarily chromatically unique. In fact, all trees are uniquely colorable, by Eisenberg [8], and yet all n-vertex trees are chromatically equivalent.

4. What properties make two graphs chromatically equivalent? Chromatic equivalence is an equivalence relation on the set of all graphs, and as such, partitions the set of all graphs into equivalence classes. The set of trees with n vertices makes up one such class. R. A. Bari [3] and L. A. Lee [18] investigate classes of graphs obtained by rotating subgraphs to arrive at new graphs, such that the chromatic polynomials of the newly obtained graphs are the same as those of the original graphs. In this report, we have shown that the class of graphs $\bar{\theta}_{2,e,f}$ where $e + f$ is a constant are chromatically equivalent, as are pairs of tree complements $n = 8,9,10,11$. Finding necessary and sufficient conditions for putting graphs into chromatic equivalence classes has proven to be as difficult as establishing equivalence classes which contain only one member, i.e. chromatically unique graphs.

Closely related to chromatic uniqueness and equivalence is the problem of chromatic distinctness. $\bar{\theta}_{d,e,f}$, $d \neq 2$, are chromatically distinct for $n \leq 17$; so are fork complements $n \geq 4$, as well as all tree complements for $n \leq 7$. As more families of graphs are analyzed, more of these results may be obtained.

5. One final problem concerns the most storage- and time-efficient implementation of algorithms for computing chromatic polynomials. As discussed in Section 2.4, the best methods now known are those that utilize Whitney's

identities in the tree basis for sparse connected graphs and in the complete graph basis for dense graphs. At this point, the cutoff for use of the tree basis seems to be $\lceil \frac{1}{2} \binom{n}{2} \rceil$ edges; if the number of edges in the graph is greater than $\lceil \frac{1}{2} \binom{n}{2} \rceil$, use the complete graph basis. To test this cutoff, we can generate random graphs and compute their chromatic polynomials relative to both bases, and compare. One must be careful here—graphs with larger automorphism groups will occur more frequently as "random" graphs than those with smaller automorphism groups. Thus we must take these automorphism groups into consideration when making the basis comparison.

It would also be interesting to generate the chromatic polynomials of all 12,346 graphs on eight vertices. This and the other research mentioned above may raise new and exciting questions, and hopefully provide a few answers, in chromatic polynomial theory.

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